THE
MATHEMATICAL GAZETTE.

EDITED BY
W. J. GREENSTREET, M.A.

WITH THE CO-OPERATION OF
F. S. MACAULAY, M.A., D.Sc., AND PROF. E. T. WHITTAKER, M.A., F.R.S.

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J. S. MACKAY.

JOHN STURGEON MACKAY was born on the 22nd of October, 1843, at Auchencairn, near Kirkcudbright, where his father, in the course of business, resided a short time, but it was in Perth that he spent his boyhood. In the Academy there he received a thorough grounding in English, Latin, Greek and Mathematics, and when in 1859 he matriculated at St. Andrews University, he was able to take full advantage of all the opportunities the University offered. While an undergraduate he read very widely in Greek and Latin literature, and had the reputation of being one of the finest classical scholars of an unusually brilliant year. At a later period he acquired a sound knowledge of French, German and Italian, his command of French, whether colloquial or literary, being exceptionally good. He completed his course and passed all the examinations for the degree of M.A. in 1863 (though he did not graduate till 1865), and in the same year he returned to his old school as Rector's assistant, remaining there till 1866, when he joined the staff of the Edinburgh Academy. In this well-known school he spent the rest of his working life, holding for the greater part of that period the position of head mathematical master.

During the early years of his residence in Edinburgh he devoted a considerable part of his leisure to the study of botany and geology, and he never lost his interest in these subjects.

Dr. Mackay's training was thus unusually broad, and when, as in his case, a liberal education was associated with a genial, upright and sympathetic nature, it is easy to understand why he was so successful as a teacher and gained both the respect and the affection of his pupils. Many of these attained distinction in various fields at home and abroad, and many proofs have been given of the deep affection they cherished for their old master; no furlough was complete that did not include a visit to the well-known room in Northumberland Street.

From his University days geometry was Dr. Mackay's favourite mathematical study; at an early date he began a careful reading of the Greek geometers in the original tongue, and as the years passed by he acquired a quite unusual familiarity with the whole range of Euclidean geometry. In the course of his researches he was led to give special attention to Pappus, and it is a matter for regret that the appearance of Hultsch's edition induced him to abandon his intention of editing the *Collections*.

It is not possible to enter here on an analysis of his contributions to the history of geometry; most of these are contained in the *Proceedings of the Edinburgh Mathematical Society*, but attention may be called to his article, *Notice sur le Journalisme mathématique en Angleterre*, contributed to the French Association for the Advancement of Science (Congrès de Besançon—1893)—an exceedingly valuable guide to one aspect of English mathematical history.

In 1884 appeared his *Elements of Eucild*; the book had a very wide circulation, and contained many novel features that were soon more or less successfully adopted by other editors. Dr. Mackay was, to say the least, not an enthusiastic supporter of the Perry movement in its bearing on the teaching of geometry; he always contended that the valuable parts of the new methods had been adopted in the better Scotch schools, and he maintained that a standard order of propositions was a practical, though not a theoretical, necessity. His *Plane Geometry*, which appeared in 1905, was adapted to the newer programmes; though he had not, I think, the same interest in preparing this later text-book, it is nevertheless a sterling piece of work, such as could only have been written by one who was master of the subject.

Other text-books from his pen are *Arithmetical Exercises* (1869) and *Arithmetic* (1899); he also contributed several articles to Chambers's *Encylopaedia* and to the *Encyclopaedia Britannica*.

Dr. Mackay was a member of the A.I.G.T. from its origin, and took a warm interest in its work.

He was one of the founders of the Edinburgh Mathematical Society, was its first President, and was for years a leading contributor to its *Proceedings*. He served for more than one period on the Council of the Royal Society of Edinburgh, and his services were of special value as a member of its Library Committee. For many years he spent the greater part of his summer holiday on the continent, and he was a well-known figure at the meetings of the French Association for the Advancement of Science. His great linguistic attainments were of much service to the Permanent International Commission for Mathematical Bibliography, of which he was a member. He also took a keen interest in the Franco-Scottish Society, and was a member both of the Council and of the Executive.

Dr. Mackay's great learning was fittingly recognised by his old University when, in 1887, it conferred on him the honorary degree of LL.D.

In 1904 he retired from active work, but he had no difficulty in finding congenial occupation; he devoted himself to his favourite studies, and was ever ready to place his great knowledge at the disposal of his friends. Until three months ago he enjoyed fairly good health, but early this year a change for the worse took place, and he passed away on the afternoon of Wednesday, 25th March.

Few men in Scotland have had a stronger influence on the development of mathematics in the secondary schools. None has equalled him in the special branch of historical learning which he made peculiarly his own, and his loss is mourned by a wide circle of pupils and friends.

G. A. Gibson.

DESARGUES' THEOREM.

In his *Foundations of Geometry*, Hilbert has shown "the impossibility of demonstrating Desargues' Theorem for plane geometry without the help of the axioms of congruence." But it must be remembered that his demonstration depends entirely on the form which he has given to the other

axioms, or definitions as I should prefer to call them, and in particular on the way he defines "straight line," namely "two distinct points always determine completely a straight line," and "any two distinct points of a straight line completely determine that line." What he has shown then is that in plane geometry this is an inadequate definition of "straight line." He has not shown that that definition cannot be amended otherwise than by the introduction of the conception of congruence, or by that of three-dimensional space.

One of the most interesting results of the theory of Order as defined by Boundaries is that it explains this apparent paradox. I propose to add to the papers already published in the *Gazette* on this theory a brief note on this point. To aid the imagination of my readers I shall not attempt to state the problem in general terms, but shall merely state it in homely geometrical language, and for the sake of brevity I shall not pretend to give formal demonstrations of each successive point in the argument, but shall leave obvious deductions to the reader.

Suppose we are given a closed surface in space; of any form whatever; say that of a pumpkin. On that with a pencil I draw any closed line (which nowhere cuts itself). This forms a boundary dividing it into two parts. For reference I give the definition of a Boundary, thus:

> A Boundary is a sub-group in an ordered group of units of thought, which divides it into two parts in such a way that:
>
> (i) It is possible to pass in review from any unit in one part to any other unit in the same part without passing any unit in the Boundary on the way.
>
> (ii) It is impossible to pass in review from any unit in one part to any unit in the other part without passing any unit in the Boundary on the way.
>
> (iii) Every unit in the Boundary is contiguous to at least one unit in each of the two parts.

Here our units of thought are points; passing in review means moving continuously in space from point to point. Thus the surface of our pumpkin is a boundary in space, dividing it into two parts, inside and outside the pumpkin. And the closed line drawn on the surface is a boundary in the surface. Any two points not in that boundary are or are not separated by it. If they are separated, we can pass from the one to the other by passing one, but not necessarily more than one, unit in the boundary on the way.

Now I draw a second closed line on the pumpkin. Let us call the first line A and the second B. I draw B so that the two points a_1 and a_2 in A are separated by B. If I pass from a_1 to a_2, therefore I pass at least one point in B, and if I complete the circuit of A and pass back to a_1, again I pass at least one other unit in B. The units common to A and B consist therefore of at least two, but not necessarily more than two, units. If they consist of two, and only two, I call that a simple intersection AB. I do not go into the general definition of simple intersections. Suffice it to say here that on our pumpkin the simple intersection of two complete lines consists of two points, and let us note the fact that any two points in either line, which are separated in that line by the simple intersection, are separated on the surface of the pumpkin by the other line, and conversely.

I now go on drawing closed lines all having the same simple intersection with each other, in the two points which constitute AB. As I draw these successive lines I name them from a catalogue of names, for which the capital letters of the alphabet may stand, for our present purpose, so that the group of closed lines through AB may be regarded as a group of the first order of lines as units of thought, collated with the catalogue group, the capital letters of the alphabet. In the limit we may conceive one, and only one, line of this group to pass through each point on the surface of the

pumpkin, with the exception of the two points AB, through which every one of the lines passes.

I now start afresh, and draw another closed line C, which also has a simple intersection with the first line A in two points AC, which are separated in A by the two points AB. And I go on drawing a group of lines, having all of them simple intersections with each other in AC, just as before, but with this additional qualification, that they all have also simple intersections with the first group of lines through AB. It can be shown that the two points in these simple intersections are in every case separated on the pumpkin by the initial line A. In the limit we may therefore conceive that any point on the pumpkin is one of a pair of points forming the simple intersection of two of the lines I have drawn, and the point may therefore be defined as the point of intersection of two named lines in the one (or the other) of the two parts into which A divides the pumpkin. If the one line is named X and the other Y, we may name the point XY, or XY', according as it is in the one part or the other with respect to A. I say that we have *catalogued* the points in the pumpkin (with the exception of the points in A itself) with respect to the *fundamental boundary* A, and I call the lines of the two groups which I have drawn on the pumpkin, *chosen lines*.

Now observe, these chosen lines possess the property that any two of them have a simple intersection. This is the property corresponding to Hilbert's axioms of connection with respect to straight lines. Is this property sufficient to determine an unique class of closed lines on the surface of the pumpkin corresponding to straight lines in geometry? No. Hilbert has shown that this is not so, and surely this is what one would expect. It would be indeed surprising if it could be shown that henceforward there was only one way in which a closed line could be drawn on the pumpkin through two given points on it which would have a simple intersection with every line already drawn. Since every point on the pumpkin is now named, and so can be uniquely identified, it must theoretically be possible, if any such unique class of closed lines exists, to deduce the names of all points on any one of them from the given names of two. There must therefore be some rule of thumb by which lines of this unique class are defined. This rule of thumb may be stated as follows:

> By passing in review the points of any *unique line* we uniquely collate the chosen lines in the two sets which intersect in them; the fundamental boundary being collated with itself in the two sets.

Now I have shown that unique collation can actually be carried out, by means of a numerical catalogue. It is therefore possible, in theory at all events, actually to draw any number of unique lines. We may, for example, conceive a whole group consisting of all the unique lines through the simple intersection of the chosen line B of the first set, and the chosen line C of the second. We may name these unique lines; say after the names of the chosen lines of the first set in which they intersect a given chosen line, say P, of the second set. We may now regard the unique lines of this group as the chosen lines in a fresh catalogue, the fundamental boundary of which is the chosen line B of the first set, and the compound names of points in this new catalogue will consist of names from this new set, combined with names from the first of the original sets of chosen lines. And it can be demonstrated that any line which is an unique line in the old catalogue will also be unique with respect to the new one. Thus the chosen lines of the second group will be unique lines with respect to the new catalogue, it being only permissible to exercise the degree of freedom implied in calling the lines "chosen" with respect to two sets of lines, all others being thereafter uniquely determined.

From this way of looking at it, it will, I hope, become evident to the

reader why it is that Hilbert's axioms, without his axiom of congruence, were insufficient to define "straight line." To show that my definition of unique line is not equally defective, it is enough, as against Hilbert at all events, to deduce from it Desargues' Theorem, which may be done as follows.

Representing unique collations by schemes, we represent an unique line by a scheme of two lines, in which the names of the fundamental boundary in each set are collated together, and therefore appear one above the other. It is convenient to call the fundamental boundary A in each set, so that the two A's appear in the scheme in the same column. The two names in any other column of the scheme represent together a point in the line. If we are collating together the points in various lines, we write their schemes one under the other, but there is no point in writing both lines of each scheme; there is no loss of generality in supposing that, the collation of the points being unique, it is effected by collating one part of the name of each identically, and the second part uniquely. One line in the scheme will therefore stand for the first part of the names of the points in all the unique lines of the problem, and each unique line of the problem will have in addition a line in the scheme containing the second parts of the names of its points. In the first line of the scheme, that is, for the first parts of the names of points, I use capital letters, otherwise I use small letters to denote chosen lines or second parts of names of points. The fundamental boundary is, however, denoted by A in both cases. For brevity I use the letter U for "unique line."

Suppose then we are given two sets of three points each whose catalogue names are

$$\begin{matrix}(Bb) & (Cc) & (Dd),\\ (Pp) & (Qq) & (Rr).\end{matrix}$$

And let the U's $\begin{matrix}CD \\ DB \\ BC\end{matrix}$ and $\begin{matrix}QR \\ RP \\ PQ\end{matrix}$ intersect in $\begin{matrix}(Xx),\\ (Yy),\\ (Zz).\end{matrix}$

(I think this notation is obvious.) That is, we are given the scheme

$$\left.\begin{array}{cccccccccc} A & B & C & D & P & Q & R & X & Y & Z \\ A & & c & d & & & & (x) & & \\ A & & & & & q & r & (x) & & \\ A & b & & d & & & & & (y) & \\ A & & & & p & & r & & (y) & \\ A & b & c & & & & & & & (z) \\ A & & & & p & q & & & & (z) \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

(Here we see that x, y, and z have to be deduced from the data of the problem.)

The condition that the points X, Y, and Z (*i.e.* the points Xx, etc.) shall be collinear is

$$\left.\begin{array}{cccc} A & X & Y & Z \\ (A) & x & y & z \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

The condition that BP, CQ, DR, shall be concurrent (in O) is

$$\left.\begin{array}{cccccccc} A & B & C & D & P & Q & R & O \\ A & b & & & p & & & (o) \\ A & & c & & & q & & (o) \\ A & & & d & & & r & (o) \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Desargues' Theorem is that if (3) is true, then (2) is also, and conversely.

I. Let it be granted that (3) is true.

Then, to simplify the discussion, we may assume that the origin of the first set of C's (chosen lines) is in the unique line XY, so that Y is the same

name as X, while the origin of the second set of C's is (Oo), so that p is the same as b, q as c, and r as d. Scheme (1) may therefore be written:

(1)	A	B	C	D	P	Q	R	X
(2)	A		c	d				(x)
(3)	A					c	d	(x)
(4)	A	b	$(-)$	d				(y)
(5)	A				b	$(-)$	d	(y)
(6)	A	b	c					$(+)$
(7)	A				b	c		$(+)$

.....(4)

Here the $(-)$ in line (4) represents the name deduced by the unique collation as collated with C in line (1), and therefore with the c in line (2), when A, d, y, are collated respectively with A, d, x. But these three names are so collated in lines (5) and (3). Hence the same name $(-)$ comes in line (5) under the c in line (3).

Again, $(+)$ in line (6) represents the name which comes in that line under X in line (1), and is therefore collated with (y) in (4), when A, b, c, are collated with A, b, $(-)$, respectively. But these names are so collated in lines (7) and (5), so that under (y) in (5) we must write $(+)$ in (7).

It follows that (BC) and (PQ) intersect in $(X+)$; that is, in a point collinear with (Xx) and (Yy), which is now called Xy.

II. Let it be granted that (2) is true.

Then, to simplify the discussion we may assume that the origin of the first set is in the U represented by (2), which is in fact the chosen group X of the first set, while the origin of the second set is the intersection of PB and CQ, so that p is the same as b and q as c. Scheme (1) may therefore be written:

(1)	A	B	C	D	P	Q	R	X
(2)	A		c	d				(x)
(3)	A					c	r	(x)
(4)	A	b	$(-)$	d				(y)
(5)	A				b	$(-)$	r	(y)
(6)	A	b	c					(z)
(7)	A				b	c		(z)

.....(5)

Here the $(-)$ in line (4) represents the name which comes under the C in line (1), and which therefore is collated with the c in line (6), when A, b, y are respectively collated with A, b, z. But these names are so collated in lines (5) and (7); so that in line (5) we can write $(-)$ over the c in line (7).

It follows that the collation of line (2) with line (4), and of line (3) with line (5) might either of them be regarded as having been determined by collating A, c, x, with A, $(-)$, y, respectively; and the collations of these pairs of lines must therefore be the same throughout, and in both cases the identically collated names must be the same. The name A is collated identically in both cases, but in the one case d, and in the other r, is also identically collated. Therefore r must stand for the same name as d; in other words, (Dd) and (Rr) are on the same chosen group of the second set, and BP, CQ, and DR are concurrent. Q.E.D.

EDWARD T. DIXON.

THE CALCULUS AS AN ITEM IN SCHOOL MATHEMATICS.

(*Continued from* p. 246.)

PASSING very briefly here, but by no means very briefly* in the classroom, over this treatment of velocity, a few words may be given to the problem of the tangent.

* Time may be saved if some familiarity with the notion of a "rate" has been gained by arithmetical problems on wages per hour or per piece, railway fare per mile or per hour on different railway journeys,—and other simple statistical investigations.

The Greeks studied a small number of curves—each defined by some characteristic geometrical property. They devised a separate geometrical construction for drawing a tangent to each separate curve.

Descartes, by introducing the notion of an equation as representing a curve, made it clear that there are "ever so many" curves, and it became evident that some general method of tangents—a method which could be applied irrespective of the geometrical peculiarities of an individual curve—was urgently required; for any man could invent curves as fast as he could write down equations.

The definition of a tangent as the limit of the chord PQ which revolves round P, as Q passes from one side to the other of P, was in substance given by Fermat.

Armed with the equations

$$\frac{\Delta y}{\Delta x} = \tan PKx,$$

$$\frac{dy}{dx} = \tan PTx,$$

the pupil is in a position to investigate tangent properties of the curves defined by equations of the second and third degrees.

It is suggested that the pupil should make, largely by examples, a somewhat careful study of

$$y = ax^2 + bx + c,$$

$$y = ax^3 + bx^2 + cx + d,$$

as to the direction and slope of tangents and simple tangent properties.

"Confidence is a plant of slow growth." We want the pupil to feel that he can tackle any simple question of this kind with certainty. A boy who *knows* that he can solve *any* quadratic has a comfortable piece of knowledge.

The fact that the tangent to a cubic meets the curve again in *one* point is perhaps *the* most characteristic geometric feature of the plane cubic curve. It supplies plenty of algebraic drill and neat results.

That the derived function exists apart from all question of proceeding to limits is a third important idea, most easily obtained through rearranging a quadratic, cubic, or rational polynomial in $x+h$ in powers of x, with the corresponding graphical illustration.

If we take, for instance, the function x^3, we have

$$(x+\Delta x)^3 = x^3 + 3x^2\Delta x + 3x \,.\, \Delta x \,.\, \Delta x + (\Delta x)^3.$$

The facts that $$\Delta(x^3) = 3x^2\Delta x,$$

with a degree of approximation depending on the smallness of Δx, and that the function $3x^2$ is perfectly free from all taint of approximation, are grasped in more assured fashion.

The student has now a thorough grip of the meaning of a differential coefficient, and he can differentiate

$$x^2,\ x^3,\ x^4,$$

their sum and difference, and also

$$ax^2,\ ax^3,\ ax^4.$$

The first formal piece of bookwork should be the differentiation of a product uv, u, v being expressions which we know how to differentiate, and from this we deduce the differential coefficient of x^n, where n is a positive integer.

It is noteworthy that it is much easier to differentiate x^n if we use the old-fashioned notation for x^n.

Thus
$$\frac{dx^2}{dx}=\frac{d(xx)}{dx} \quad =1\times x+x\times 1=2x,$$

$$\frac{d(x^3)}{dx}=\frac{d(xxx)}{dx} \quad \begin{aligned} &= 1\times x\times x \\ &\quad +x\times 1\times x \\ &\quad +x\times x\times 1 \quad =3x^2, \end{aligned}$$

$$\frac{dx^4}{dx}=\frac{d(xxxx)}{dx} \quad \begin{aligned} &= 1\ x\ x\ x \\ &\quad +x\ 1\ x\ x \\ &\quad +x\ x\ 1\ x \\ &\quad +x\ x\ x\ 1 \quad =4x^3, \end{aligned}$$

and the induction for x^n (n a $+^{ve}$ integer) is obvious, and may be formally developed thus:

$$y=x^n=xx^{n-1},$$

$$\frac{dy}{dx}=1\times x^{n-1}+x.(n-1)x^{n-2}$$
$$=nx^{n-1}.$$

More elegantly—but less easily for youngsters—we have, as Mr. Carson indicated at a recent meeting,

$$\frac{1}{uv}\frac{d(uv)}{dx}=\frac{(uv)'}{uv}=\frac{u'}{u}+\frac{v'}{v}.$$

and
$$\frac{(uvw\ldots)'}{(uvw\ldots)}=\frac{u'}{u}+\frac{v'}{v}+\ldots\frac{w'}{w},$$

so if
$$u=v=w=x,$$

$$\frac{(x^n)'}{x^n}=\frac{n}{x}.$$

Our second formal theorem is that

$$\frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx},$$

and we use it to find the differential coefficient of $x^{\frac{p}{q}}$; p, q and n being positive integers.

And now some practical problems on maxima and minima, mechanics, physics and geometry, are within our reach.

When we come to the Trigonometrical Functions, the time expended on the careful analysis of the idea of velocity begins to earn a dividend.

Consider a point moving round a circle of 1 foot radius at a speed of 1 foot per second. If the point is at P, t seconds after it left A, the angle AOP is t radians.

The velocity of P resolved along Oy is $1\times\cos t$.

But the velocity of P along Oy is $\frac{dPM}{dt}=\frac{d\sin t}{dt}$, the rate at which its distance from Ox is growing, so $\frac{d\sin t}{dt}=\cos t$.

Also the velocity of P resolved along Ox is $\sin t$ towards O, and is also $\frac{dOM}{dt}$ or $\frac{d\cos t}{dt}$ away from O, so $\frac{d\cos t}{dt}=-\sin t$.

In obtaining the important series for $\sin x$ and $\cos x$ we must consider carefully what weight we mean to allow to the claims of mathematical

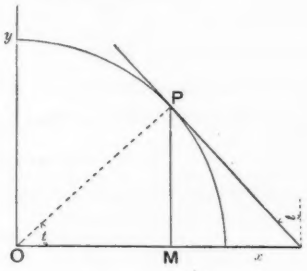
rigour. I must confess that personally I cannot bring myself to doubt that the natural method of obtaining the expansions is to assume boldly

$$\sin\theta = A_0 + A_1\theta + A_2\theta^2. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Put $\theta=0$. A_0 must vanish. Differentiate the original equation, obtaining

$$\cos\theta = A_1 + 2A_2\theta + 3A_3\theta^2. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Put $\theta=0$ and A_1 must be 1, and so on.



Now, however, with an eye to the future of the better pupils, is the time to insinuate a doubt.

The simple observation that when n is large and x is small, nx^{n-1} is ever so much larger than x^n suffices. Each time we differentiate we increase the relative weight of the remoter terms. Are the terms which are negligible in (1) also negligible in (2) and in (3), (4), (5), ...?

This is about as much as we can do for the rigorist at present—to make pupils see that there is a point for discussion and that a doubt is not necessarily a mere quibble. To take a legal illustration. When the layman learns that twenty-four judges were sorely puzzled to determine whether a man who kept a sovereign which was handed to him in mistake for a shilling, could be convicted of stealing it, he laughs at the lawyers' quirk. But let the laughter become personally involved—of course in some more creditable manner. Suppose that he was to get a legacy if Mary Doe died "unmarried," mark the word, and the question is whether the word "unmarried" meant "without ever having been married"—that is, a spinster, or "without leaving a husband surviving her." In such a case our champion of simple common-sense will display a creditable talent for inventing quibbles for himself. The analogy is a fair one.

Do not give the proofs based on the sign of the successive differential coefficients of

$$\sin\theta - \theta + \frac{\theta^3}{3!}, \text{ and of } \sin\theta - \theta + \frac{\theta^3}{3!} - \frac{\theta^5}{5!},$$

etc., until you are quite sure that they will be appreciated as distinctly soothing. Do not let us jump before we come to the fence.

There is of course a more serious objection to starting with any such proof. From the point of view of a boy who is totally unacquainted with the expansion, the "proof" is a most repulsive piece of trickery. Why on earth should you write down $\sin\theta - \theta + \frac{\theta^3}{6}$ rather than any of a few million other expressions?

INTEGRATION.

As regards integration—here again a personal conviction is expressed by saying that we should start by regarding integration as the converse of differentiation.

The difficulty is the transition from this point of view to that of summation.

We have to realise, as Dr. Sheppard remarked to me, that the received notation bears two distinct interpretations:

$$\frac{x^3}{3} = \int \mathbf{x}^2 dx,$$

or $$\int_0^x \mathbf{x}^2 \mathbf{dx}.$$

In the first instance we can avoid dwelling on the idea of summation. We show in the familar manner that if $y=x^2$, then $\frac{\Delta A}{\Delta x}$ lies between x^2 and $(x+\Delta x)^2$, and hence that

$$\frac{dA}{dx} = x^2. \quad \text{.....(1)}$$

The points to make are (1) that this statement holds good for *any* position of the ordinate; and secondly, that the equation

$$A = \int \mathbf{x}^2 dx = \frac{x^3}{3} + C$$

is only a translation from one symbolic language into another, and asserts neither more or less than (1).

That is to say, since we have obtained all our information from the right-hand end of the diagrams, the area from an undetermined starting-point up to

$$x=a \text{ is } \frac{a^3}{3} + C,$$

$$x=b \text{ is } \frac{b^3}{3} + C,$$

and so area between these limits $= \frac{a^3 - b^3}{3}$.

Much trouble later may be saved by this attention from the very first to the inevitableness of the arbitrary constant.

For practical purposes, the one thing needful is to realise that if we can divide the subject-matter into suitable elements, and write down the contribution of one such element to the required result correct to the first order of small quantities, we have a routine process for finding—at any rate in form—the total of all the contributions.

The fact that the contribution of the element need only be correct to the *first* order of small quantities must be grasped. Many of the failures of beginners are due to a creditable striving after greater accuracy.

Further, at any rate at the outset, every step in the argument must be set out with care, and the pupil must not be confronted with mathematical and physical difficulties simultaneously.

It should perhaps be added that though for convenience in writing this paper, Integration has been kept separate from Differentiation, yet no doubt Integration as the converse of Differentiation and the application to areas, volumes, etc., should come in early, but not—as perhaps is sometimes the case—too early. Exceedingly respectable students will sometimes formally differentiate when they mean to integrate and *vice versa*, which suggests that the treatment of these subjects side by side may be overdone.

On the other hand, lads of seventeen and eighteen, dealing with work or distance diagrams, have no business to be counting squares in order to find the area of a trapezium. Yet this is not uncommon.

The opportunity should not be lost of calling attention to the Integraph of Professors Abdank Abakanowicz* and C. V. Boys, and to some of those primitive planimeters which carry out the summation of $y\Delta x$ in so concrete and obvious a manner.

The wickedness of ignoring the evolution of the planimeter and beginning the subject with a description of Amsler's planimeter, in its final form, is almost beyond belief.

The very serious question of the boundary between a school and a university course is involved in the remaining question, the differentiation of a logarithm and the introduction of e.

The natural sequence would seem to be to try to differentiate $\log x$.

The limit $\left(1+\frac{1}{n}\right)^n$ as $n \to$ infinity forces itself into notice, though considerations of an automatic pass-book crediting interest all the time may be helpful.

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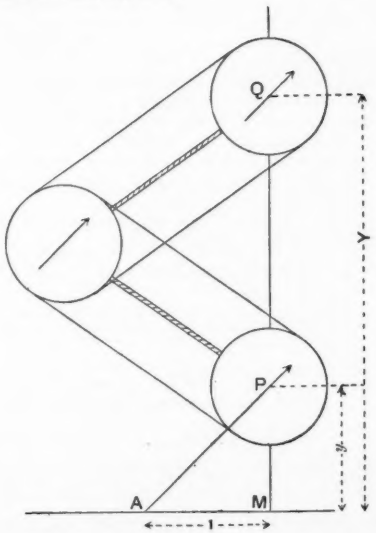
*Abdank Abakanowicz, *Les Intégraphes*. A tracing point P, sliding on the tiller AP, which pivots on the fixed point A, is made to describe the curve

$$y = f(x),$$

the distance AM being constant, say unity.



By various ingenious mechanical means, *e.g.* three equal pulleys with tightly stretched belts, a wheel Q is kept parallel to AP, and with its point of contact with the paper on MP produced. The wheel traces out on the paper a curve whose slope

$$\frac{dY}{dx} = \tan PAM = y = f(x),$$

$$Y = \int f(x)\,dx.$$

The many ingenious attempts to introduce e by a graphical investigation of the curve in which

$$\frac{dy}{dx}=y$$

are all to my mind stamped with the brand of artificiality, and to re-define the logarithmic function as the value of $\int_1^x \frac{1}{x}dx$ is of doubtful expediency, though it is of course easy to show that $\frac{d\log x}{dx}$ must be of the form $\frac{A}{x}$.

No doubt, however, a reluctance to follow a new path may be due to mere conservatism. There are few things more difficult than to compare impartially a proposed new proof with one that has been familiar for years.

But if the traditional method is retained, grave difficulties present themselves.

The schoolboy is taking an explorer's journey through a country. He notes of a bridge that probably large animals would be unwilling to cross it, but he does not stay to strengthen it.

When the university stage is reached and it is necessary to survey the details of the country with more precision trouble is apt to arise.

A boy who thinks he has disposed of the exponential theorem does not always see why he should return to it.

Some years ago—in a paper read to the Mathematical Association (*Gazette*, vol. iii. p. 76)—the remark was made that "the due discrimination on broad lines between the work proper to a school and the work proper to a university is a problem that calls for our most serious attention—the solution of which will dispose of many of our difficulties." This opinion is shared by Dr. Filon, who says (in a letter quoted with his permission): "The time is coming when a serious effort will have to be made to co-ordinate school and university education. . . . Boys come up to the university who cannot prove . . . the binomial theorem for a fractional index. . . . When they see that the university teacher does not accept their 'proofs' and puts them back to work which they imagine they have done before, they sometimes turn sullen and sour, and that may be an end of them as far as mathematics is concerned."

What is the solution? We must help the schoolboy to see distinctly that the bridge is weak, without stopping to mend it.

Thus, when we start to consider the limit $\left(1+\frac{1}{n}\right)^n$ and write as usual

$$\left(1+\frac{1}{n}\right)^n = 1+n.\frac{1}{n}\ldots+\frac{n.n-1\ldots n-r+1}{1.2\ldots.r}\frac{1}{n^r}\ldots$$

$$=1+1\ldots+\frac{1.\left(1-\frac{1}{n}\right)\ldots.\left(1-\frac{r-1}{n}\right)}{1.2\ldots.r}\ldots$$

$$=u_0+u_1\ldots+u_r\ldots,$$

he should note firstly that a race is taking place.

As r increases, the term u_r is becoming insignificant in the fullest sense of that word, and the statement that $1-\frac{r-1}{n}$ is practically unity is ceasing to be true.

A race is always interesting.

Of course, however, the real difficulty has not yet been reached. We have assumed that n is a positive integer, and the most fireproof investigation of the limit when the positive integer n increases indefinitely will not help us to find the limit when n increases continuously. This, too, may be made quite clear to a boy of sixteen.

What is the limit of $\sin\frac{1}{x}$ as x becomes "ever so small." The answer must be "it all depends."

As n, a positive integer, $\to$ infinity,

if $x=\frac{1}{2n\pi}$, $\sin\frac{1}{x}$ is and remains 0;

if $x=\frac{1}{2n\pi+\alpha}$, $\sin\frac{1}{x}$ is and remains $\sin\alpha$;

if $x=\frac{1}{2\sqrt{2}n\pi}$, ? ? ?

As n, unrestricted in form, tends continuously to infinity,

if $x=\frac{1}{2n\pi}$, $\sin\frac{1}{x}$? ? ?

The queries must remain unanswered.

A boy who passed over to the university at this stage would at any rate know why, now that the time had come to put away childish things, it was necessary to reinvestigate the limit of $\left(1+\frac{1}{n}\right)^n$ as $n\to\infty$ continuously.

What has been gained.

A boy who had gone through the course which has been briefly sketched out would have some power over simple practical problems and some notion, amounting, it may be hoped, to sympathetic insight, into the general nature of mathematical precision.

A considerable amount of time will have been incidentally saved from the traditional school courses in mechanics, conics, and mensuration. No longer need the principle of virtual work be so deeply mysterious—no longer must centres of gravity and moments of inertia* be either assumed or found by tedious applications of "indivisibles."

What has been lost.

The subject has come into a lower plane. Logical discussions are out of place. It may be that the notion of a limit is like a boy's conscience—the less you talk to him about it the better. Careful investigation alone can show how far a reasonable standard of mathematical precision can be really attained, appreciated, demanded, and not merely endured by the average boy.

We all remember M. Taine's statement, that after a conscientious study of the history and institutions of France he felt sure of this proposition only, that a civilised state is a very complex organism. To write a paper like this and to review for that purpose the thoughts of twenty years spent in trying to teach the calculus is to be reminded continually how complex is the problem of the teacher—how narrow the line between verbosity and

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*May yet one more digression in this discursive paper be permitted? Cannot the abominable phrase "moment of inertia" be stamped out? When applied, as it often is, to an area it is difficult to see a single argument in favour of its retention.

If "swing radius" or "radius of gyration," with its perfectly definite physical meaning, is not acceptable as a substitute, cannot we agree to adopt the term "second moment about an axis" as an equivalent?

The mathematical analogy between the first and second moments for the case of n unit particles is illuminating, and the colourless character of the name renders its application to areas, volumes, weights and masses unobjectionable.

For n unit particles, and an axis Oy,

First moment $M_1=n\bar{x}=x_1+x_2+\ldots+x_n$.

Second moment $M_2=nk^2=x_1^2+x_2^2+\ldots+x_n^2$.

(Note added, 1913.) I was very glad to hear from conversation in 1912 with Professor E. V. Huntington that this is also his opinion.

obscurity—how easily a teacher may aim too high, and in striving after mathematical elegance, precision, and generality become unintelligible—how easily a teacher may aim too low and sink to the mean and dull through anxiety to keep well within the comprehension of his pupils—that our task of trying to teach people to think by showing them methods of avoiding laborious thinking is not a very simple one, that it is wrong always to puzzle your pupils and also wrong never to puzzle them. Therefore, though some dogmatic statements have been made, I would end by disclaiming any intention to lay down as law any proposition save one—that this problem must be approached in a broad-minded way. C. S. JACKSON.

THE TEACHING OF EASY CALCULUS TO BOYS.

(*Continued from* p. 208.)

ANOTHER useful application is that the parabola, in the form $y=-k.x^2$, represents the path of a projectile. The boys at this stage have already dealt with equations of the form $y=\pm(x-a)(x-b)$, and are able to find the points where the curve is cut by the x or the y-axis.

Hence, in the equation $y=-(t-a)(t-b)$, since y denotes the height of a point on the curve above the x-axis, $\frac{dy}{dt}$ gives the vertical velocity at time t.

The velocity curve $\frac{dy}{dt}$ against t is drawn, and the question arises, when is the velocity zero? or where is the highest point of the curve? and we enter upon the subject of Maximum and Minimum Values.

This is first discussed as finding the highest or the lowest point of a given parabola. A parabola such as $2y=-(x+2)(x-3)$ is drawn, and several successive values of $\frac{\delta y}{\delta x}$ are found and compared. It is thus shown that δy δy gets smaller and smaller until at a certain point $\delta y=0$, after which it becomes negative.

Plenty of illustrations will occur here (*e.g.* a man walking over a hill top) to show that at the highest point of the curve the direction is for an instant horizontal. A reference to the mountain railway at the White City never fails to arouse interest, and the boys soon appreciate what the change in $\frac{dy}{dx}$ means, and there is little difficulty in getting them to see that when they are at the top of the hill, then $\frac{dy}{dx}=0$.

(The other case—the minimum point—is reserved for a subsequent lesson.)

Exercises are then worked. Find the coordinates of the highest point of the curve of which the general equation is

$$y=a+bx+cx^2.$$

When this is fairly well grasped a parabola of the form $y=+kx^2$ is taken, and by following similar lines to those in the previous discussion, the minimum point is found, and some exercises of this type are worked.

Then comes the question: Since the equation $\frac{dy}{dx}=0$ gives both the highest as well as the lowest points of the curves, and the process of working is similar in each case, how shall we distinguish between them without the trouble of graphing the curve. Here again the car on the mountain railway will carry us over these points. What is the slope of the car when

$$\frac{dy}{dx} \text{ is } +, \text{ when } \frac{dy}{dx} \text{ is } 0, \text{ and when } \frac{dy}{dx} \text{ is } -?$$

and we arrive at the rule that in passing over a maximum point the order of change in the value of $\frac{dy}{dx}$ is $+, 0, -$, while in passing a minimum point the order of change is $-, 0, +$.

Hence we find, by trial, the value of $\frac{dy}{dx}$ when x is a little less than, and again when x is a little more than the value which makes $\frac{dy}{dx}=0$.

Note.—I have not found the method of taking the second differential very successful with boys. There seems to be a lack of suitable illustrations.

This subject opens up a wide field for exercises of a problem kind, and which can be given at frequent intervals throughout the remainder of the course. In these problems the real difficulty is to get the boys to state the conditions as a mathematical equation.

The next figure that we study is the circle.

The same method of investigation that has been adopted for the parabola is followed here. This being taken for granted, it will not be necessary for me now to dwell upon those parts which are similar to the preceding work, but it will be sufficient to show how new methods are introduced, and any other points of special interest.

As before, new mathematical operations are taught as the need for them arises.

The boys will already know the equation of the circle, and a first exercise is to find the value of $\frac{\delta y}{\delta x}$. This is done by taking two points,

$$(x-\delta x_1), (y-\delta y_1) \text{ and } (x+\delta x_1), (y+\delta y_2),$$

one on each side of the point (x, y) on the circle, and we have

$$(y-\delta y_1)^2=r^2-(x-\delta x_1)^2,$$

$$(y+\delta y_2)^2=r^2-(x+\delta x_1)^2,$$

from which we get

$$\frac{\delta y}{\delta x}=-\frac{x}{y}.$$

This result is verified for various points on the circumference.

(2) The value of $\frac{dy}{dx}$ may be also found by the method of substitution, *i.e.* by letting $y^2=z$; in fact, this problem becomes the peg upon which to hang this method of substitution.

No particular difficulty has been found in getting boys to grasp this method. First find $\frac{dy}{dz}$, then $\frac{dz}{dx}$; dz is the connecting link. Plenty of examples for practice should be worked.

The area of a circle is found by assuming the circle to be divided into a large number of sectors. The method by division into strips parallel to the y-axis may be deferred for the present if they are not sufficiently advanced with their Trigonometry.

It is well now to take the volume of the sphere. Consider the hemisphere to be generated by the rotation of a quadrant of the circle about the x-axis, and we have

$$\text{the volume of hemisphere}=\pi\int_0^{\frac{\pi}{2}} y^2dx,$$

and by varying the limits the volume of a section of the sphere is obtained.

The next investigation is to study by what law the value of the sine of an

angle changes as the angle itself changes, that is, to find the value of the ratio $\frac{\delta \sin \theta}{\delta \theta}$ for any given value of θ.

The boys will be now quite familiar with the measurement of angles in radians.

(1) The first method is the usual geometrical method, from which it is made clear that the value of $\frac{\delta \sin \theta}{\delta \theta}$ approximates to $\cos \theta$ as $\delta\theta$ diminishes.

(2) The second method is as follows:

Let
$$y - \delta y_1 = \sin(\theta - \delta\theta_1),$$
$$y + \delta y_1 = \sin(\theta + \delta\theta_1),$$

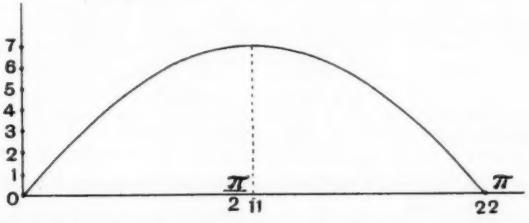
and we obtain
$$\frac{\delta y}{\delta \theta} = \cos\theta \left(\frac{\sin \delta\theta_1}{\delta\theta_1} \right).$$

Here $\frac{\sin \delta\theta_1}{\delta\theta_1}$ is the factor causing an error due to the size of $\delta\theta$.

When $\delta\theta_1 = \frac{\pi}{22}$, then $\frac{\sin \delta\theta_1}{\delta\theta_1} = \cdot 997$ approximately, and $\therefore\ 7\delta y = \cdot 997 \cos\theta$.

This equation provides us with a good practical method of differentiating graphically the sine curve.

Take $\delta\theta = \frac{\pi}{22}$, *i.e.* divide π radians on the x-axis into 22 equal parts. Then multiply each δy by 7, and the derived curve is very nearly of the same amplitude as the original sine curve.



The value of $\frac{\delta \cos\theta}{\delta\theta}$ is obtained by similar methods.

Each of these results is verified numerically with the help of tables.

The reverse exercises are then worked as graphic integration, and the boys should have little difficulty in finding out the method unaided.

Exercises follow upon the area of the sine curve between given limits.

The next step is to differentiate the forms:
$$y = A \sin\theta,$$
$$y = \sin(n\theta),$$
$$y = \sin(\theta + \alpha),$$

and
$$y = A \sin\{(n\theta) + \alpha\}.$$

By this time a greater variety of exercises is possible, many of which can be set in problem form.

The area of a circle in the form
$$\text{Area} = \int_0^r \sqrt{r^2 - x^2}\, dx$$

may be now found by changing the integral to the form

$$r^2\int_0^{\frac{\pi}{2}}\cos^2\theta\,d\theta\,;$$

and this serves as an introduction to integration by parts and the differentiation of a product; but this part may be left to a later stage, or in some classes is omitted altogether.

At somewhere about this stage of the work exercises are brought in on calculating the value of the sine and cosine of a given angle, and also calculating the value of π.

The boys have been using for some time tables of sines, and they are really interested in the question—How were these tables first got out? and they are keen on being able to calculate the value of sine $\frac{1}{2}$ radian for themselves without any help from books.

The method is on these lines. They are reminded that

$y=a+bx$ represents a straight line;

$y=a+bx+cx^2$ represents a parabola;

$y=a+bx+cx^2+dx^3$ represents a cubic curve, and so on;

and then I tell them that it has been proved that certain curves can be represented by equations of this form; that in some cases a very large number of terms must be taken; that the subject is dealt with in books on the Calculus, under Taylor's theorem, and that $y=\sin\theta$ is one of these curves. Then

$$\sin\theta \text{ is assumed} = a+b\theta+c\theta^2+d\theta^3, \text{ etc.},$$

and by alternately making $\theta=0$, and, by differentiating, the values of a, b, c, d, etc., are obtained.

Then by taking $\theta=\cdot5$ in the series obtained, the value of sin ·5 radian is obtained.

The series for $\cos\theta$ is worked out in the same way; the value of $\frac{\pi}{4}$ has also been calculated on similar lines.

The boys need a fair amount of help to get through these the first time, then the formal workings are entered in their note books for reference, and these provide material for further exercises in calculation.

In dealing with logarithms and the hyperbola, the use of the Calculus plays only a small, although an important part of the whole work, and time will permit me to indicate only the nature of the work done. The curve $y=e^x$ is differentiated both graphically and algebraically, and the nature of the error due to the finite size of δx is pointed out.

The curve $y=\log_e x$ is differentiated, and the curve $\frac{dy}{dx}=\frac{1}{x}$ is integrated graphically, and the area of portions of the rectangular hyperbola between given ordinates found.

The above is a brief outline of the nature and scope of the Calculus work of our third year course, and in conclusion I would repeat that the work is of the simplest kind, and does not require much skill in algebraic manipulation, the object aimed at being to give the boys, before leaving school, a start in the subject, with an appreciation of its value, so that they will continue the study at evening classes.

I would like to add that it is now seven years since we first introduced this subject in the school. During this time the course has been modified, and many new methods tried, and I am glad that your committee has given me the opportunity to bring it before you.

In spite of the criticism which may be urged against this method or that method, I hope that it will be felt generally that this branch of mathematics ought to find a place in the curriculum of every secondary school.

W. KNOWLES.

THE ANHARMONIC RATIO OF FOUR POINTS IN SPACE AND THE DISCORD OF TWO PAIRS OF POINTS.

§ 1. *The Anharmonic Ratio expressed in terms of Squared Distances.*

In Cremona's notation, the symbol (AB, CD) stands for the quantity

$$\lambda \equiv \frac{AC}{BC} : \frac{AD}{BD}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

where $ABCD$ are four collinear points, and A, B and C, D are conjugate pairs.

It is easy to show that

$$\lambda = \frac{AC \,.\, BD}{BC \,.\, AD} = \frac{AD^2 + BC^2 - AB^2 - CD^2}{AC^2 + BD^2 - AB^2 - CD^2}. \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

§ 2. *Anharmonic Ratio of Four Points in Space.*

The expression (2) has a meaning when A, B, C, D are not collinear points, and even when they are not coplanar. Let us take it as defining the anharmonic ratio of any four points in space, in the order indicated by the symbol (AB, CD).

To find a geometrical interpretation for the numerator of (2), let us take M as the mid-point of AB, and b, d the projections of B, D on AC.

Then
$$AD^2 - CD^2 = Ad^2 - Cd^2 = 2AC \,.\, Md,$$
$$BC^2 - AB^2 = bC^2 - Ab^2 = 2CA \,.\, Mb :$$
$$\therefore\ AD^2 + BC^2 - AB^2 - CD^2 = 2AC(Md + bM)$$
$$= 2AC \,.\, bd. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

This is twice the "scalar product" of AC and BD, *i.e.* twice the product of the length of one and the length of the projection on it of the other, regard being paid to sense.

Thus, for any four points A, B, C, D in space, we may define the anharmonic ratio (AB, CD) to be the ratio of the scalar product of AC, BD to the scalar product of BC, AD. This, of course, includes the ordinary definition as a particular case.

It follows at once that the anharmonic ratio of four points is unaltered, if we change the order of both pairs of conjugates, or if we interchange the pairs of conjugates, or if we do both.

By inspection of the formula (2), likewise, these theorems are at once evident, and we easily deduce from (2) the well-known relations between the different values of the anharmonic ratio, λ, $\frac{1}{\lambda}$, $1-\lambda$, $\frac{1}{1-\lambda}$, $\frac{\lambda}{\lambda-1}$, $\frac{\lambda-1}{\lambda}$, which are got by changing the order of the points in all possible ways.

§ 3. *The Harmonic Tetrastigm.*

Corresponding to the harmonic range of four collinear points, we might define a *harmonic* tetrastigm as consisting of two pairs of points, A, B and C, D, such that $(AB, CD) = -1$, where (AB, CD) has its generalised definition.

Thus AB, CD will form a harmonic tetrastigm (or quadrangle), if $ABCD$ be any four points in space satisfying the condition that the sum of the scalar products of AC, BD and BC, AD should be zero.

The condition can also be written

$$AC^2 + AD^2 + BC^2 + BD^2 = 2AB^2 + 2CD^2. \quad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

The form of this equation shows at once that A and B or C and D may be interchanged, or A and B may be interchanged with C and D, or any combination of these changes may be made, without altering the equation except in form.

If we denote by M the mid-point of AB, the equation (4) can be thrown into several different forms involving only squared distances.

By a well-known geometrical theorem,

$$AC^2+BC^2=2AM^2+2MC^2,$$
$$AD^2+BD^2=2AM^2+2MD^2.$$

Hence (4) reduces to

$$4AM^2+2MC^2+2MD^2=8AM^2+2CD^2,$$

or
$$MC^2+MD^2-2AM^2=CD^2;$$

i.e. the sum of the *powers* of the points C, D with regard to the sphere on AB as diameter is equal to the square on the distance between C and D. This is a well-known form of the condition that C and D should be conjugate points with reference to that sphere, *i.e.* that C should lie on the polar plane of D and *vice versa*.

Thus, two pairs of points A, B, and C, D form a harmonic tetrastigm, if C and D are conjugate points with regard to the sphere whose diameter is AB, or if A, B are conjugate with regard to the sphere whose diameter is CD; and either of these alternative conditions includes the other.

§ 4. *The Discord of Two Pairs of Points.*

The value of the quantity

$$AC^2+AD^2+BC^2+BD^2-2AB^2-2CD^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

may be named the *Discord* of AB, CD, and denoted by $\delta(AB, CD)$. One geometrical interpretation it has is the sum of the scalar products of AC, BD and of AD, BC. It vanishes when the four points, taken in the proper order, are harmonic.

To get another interpretation, let c and d be the projections of C and D on AB; and c', d' their projections on the planes ABD, ABC respectively.

Then
$$\begin{aligned} AC^2+AD^2+BC^2+BD^2-2AB^2-2CD^2 &= Ac^2+cC^2+Ad'^2+d'D^2+Bc^2+cC^2+Bd'^2+d'D^2 \\ &\quad -2AB^2-2Cd'^2-2d'D^2 \\ &= Ac^2+Ad^2+dd'^2+2cC^2+Bc^2+Bd^2+dd'^2 \\ &\quad -2AB^2-2(Cc+dd')^2-2cd^2 \\ &= Ac^2+Ad^2+Bc^2+Bd^2-2AB^2-4Cc\,.\,dd'-2cd^2 \\ &= \delta(AB, cd)+4Cc\,.\,d'd, \end{aligned}$$

$$\delta(AB, CD)=\delta(AB, cd)+4Cc\,.\,Dd\cos\theta, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

where θ is the angle between the planes DAB and CAB.

Note that, since A, B, c, d are collinear,

$$\delta(AB, cd)=Ac\,Bd+Bc\,.\,Ad.$$

Again, taking M as before to denote the mid-point of AB, and denoting by C' the inverse of C with respect to the sphere on AB as diameter, and by N the mid-point of MC, we have

$$\begin{aligned} \delta(AB, CD) &= AC^2+BC^2+AD^2+BD^2-2AB^2-2CD^2 \\ &= 2AM^2+2MC^2+2AM^2+2MD^2-8AM^2-2CD^2 \\ &= 2MC^2-4AM^2+2MD^2-2CD^2 \\ &= 2MC^2-4MC\,.\,MC'+4MC\,.\,Nd \\ &= 2MC(2MN-2MC'+2MC\,.\,Nd) \\ &= 4MC\,.\,C'd. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7) \end{aligned}$$

Thus $\delta(AB, CD)$ = twice the rectangle contained by MC, and the distance of D from the polar plane of C with regard to the sphere whose diameter is AB, the perpendicular distance from the polar plane of C being reckoned positive when D and M are on opposite sides of it.

Certain other expressions for $\delta(AB, CD)$ in terms of the mutual distances of the points $A, B, C, D, K, L, M, P, Q, R, O$ may now be stated, K, L, M, P, Q, R being the mid-points of BC, CA, AB, AD, BD, CD respectively, and O the common mid-point of PK, QL, and RM.

$$\delta(AB, CD) = AC^2+BC^2+AD^2+BD^2-2AB^2-2CD^2$$
$$=2AM^2+2MC^2+2AM^2+2MD^2-8AM^2-2CD^2$$
$$=2(MC^2+MD^2-2AM^2-CD^2) \quad \ldots\ldots(8)$$
$$=2(2MR^2+2RC^2-2MA^2-4RC^2)$$
$$=4(MR^2-MA^2-RC^2) \quad \ldots\ldots(9)$$
$$=2(MC^2+MD^2-CD^2-2MA^2)$$
$$=4S(MC\,.\,MD)-AB^2, \quad \ldots\ldots(10)$$

where $S(MC\,.\,MD)$ stands for the scalar product of MC, MD.

Again, $OA^2+OB^2+OC^2+OD^2=2MA^2+2MO^2+2R^2+2RO^2$
$$=2MA^2+2RC^2+MR^2;$$
$$\therefore\ \delta(AB, CD)=4MR^2-4MA^2-4RC^2$$
$$=4(OA^2+OB^2+OC^2+OD^2)-12(MA^2+RC^2). \quad \ldots\ldots(11)$$

The geometrical interpretation of (8) is $\frac{1}{2}\delta(AB, CD)$ = sum of powers of C and D as to sphere (AB) diminished by square on CD. (We use the contraction sphere (AB) for the sphere whose diameter is AB.)

It follows that $\frac{1}{2}\delta(AB, CD)$ is also equal to the sum of the powers of A and B as to sphere (CD) diminished by square on AB.

Again, (9) shows that the vanishing of the discord implies that sphere (AB) and sphere (CD) are orthogonal, which, of course, is equivalent to the condition that either pair of points is conjugate with regard to the sphere whose diameter is the line joining the other pair.

Query. Might there not be some advantage in extending the definition of the *power* of a point, with regard to a sphere or circle, to the case of two spheres or circles? If we define the power of one sphere as to another, as the excess of the square on the line joining their centres over the sum of the squares on their radii, then this will include the usual definition when one sphere or circle reduces to a point. With this definition (9) would be stated thus: The discord of the two pairs of points A, B and C, D is four times the mutual power of the spheres on AB and on CD as diameters. R. F. MUIRHEAD.

A FIRST CHAPTER ON ELLIPTIC FUNCTIONS.

1. There is, of course, nothing really new to be said about the elementary theory of the Elliptic Functions sn, cn, dn; but it may be not too late in the day for a didactic note on the existence and one-formedness of these functions to have some usefulness. One reason, and perhaps the chief one, why the older analysis has fallen into disfavour in the treatment of elliptic functions is the obscurity on the subject of one-formedness which prevails in so many presentations of the early parts of the theory. Halphen's *Fonctions Elliptiques* is exceptionally valuable in that it neither avoids nor postpones the difficulty; but it is common for analysts who follow the older methods either to pass the matter over without proper emphasis, or to confess weakness by appealing to the ideas of the general theory of functions, thus encouraging a belief that the newer analysis is absolutely necessary for soundness.

We have to try to establish the existence for all values of u, real and imaginary, of functions sn u, cn u, dn u, each one-valued for a given u, which

are defined by (1) the initial values 0, 1, 1 at $u=0$, and (2) either the differential equations

$$\mathrm{sn}'u=\mathrm{cn}\,u\,\mathrm{dn}\,u,\quad \mathrm{cn}'u=-\mathrm{sn}\,u\,\mathrm{dn}\,u,\quad \mathrm{dn}'u-k^2\,\mathrm{sn}\,u\,\mathrm{cn}\,u,\quad \text{with}\quad k^2<1$$

or one of these three equations and the two first integrals

$$\mathrm{cn}^2u+\mathrm{sn}^2u=1,\quad \mathrm{dn}^2u+k^2\mathrm{sn}^2u=1.$$

The first of these two integrals tells us that cn u and sn u are the cosine and sine of some real or imaginary amplitude am u, which need not be distinguished from the result of adding to it a multiple of 2π. From these dn u is determinate as

$$\mathrm{sn}'u/\mathrm{cn}\,u,\ \text{or}\ -\mathrm{cn}'u/\mathrm{sn}\,u,\ \text{or}\ \frac{1}{i}(\mathrm{cn}'u+i\,\mathrm{sn}'u)/(\mathrm{cn}\,u+i\,\mathrm{sn}\,u),\ \textit{i.e.}\ \frac{d}{du}\mathrm{am}\,u.$$

2. We will first give a proof of the addition formulae free from all assumption as to the nature of u and v.

If sn u, cn u, dn u, sn v, cn v, dn v all exist, use of the first integrals above tells us that

$$\mathrm{cn}^2u+\mathrm{sn}^2u\,\mathrm{dn}^2v=1-k^2\mathrm{sn}^2u\,\mathrm{sn}^2v=\mathrm{cn}^2v+\mathrm{sn}^2v\,\mathrm{dn}^2u.$$

Thus the equations

$$\frac{\cos x}{\mathrm{cn}\,u}=\frac{\sin x}{\mathrm{sn}\,u\,\mathrm{dn}\,v}=\text{a definite value of }\frac{1}{\sqrt{1-k^2\mathrm{sn}^2u\,\mathrm{sn}^2v}}$$
$$=\frac{\cos y}{\mathrm{cn}\,v}=\frac{\sin y}{\mathrm{sn}\,v\,\mathrm{dn}\,u}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

are consistent for the determination of a real or imaginary x and y (but for added multiples of 2π).

From the first and last of these equalities, *i.e.*

$$\tan x=\frac{\mathrm{sn}\,u\,\mathrm{dn}\,v}{\mathrm{cn}\,u},\quad \tan y=\frac{\mathrm{sn}\,v\,\mathrm{dn}\,u}{\mathrm{cn}\,v},$$

we at once obtain

$$dx+dy=\frac{\mathrm{dn}\,u\,\mathrm{dn}\,v-k^2\,\mathrm{sn}\,u\,\mathrm{sn}\,v\,\mathrm{cn}\,u\,\mathrm{cn}\,v}{1-k^2\mathrm{sn}^2u\,\mathrm{sn}^2v}(du+dv).\quad\ldots\ldots\ldots\ldots(2)$$

Thus $du+dv=0$ necessitates $dx+dy=0$; so that to be given $u+v$ constant is to be given $x+y$ constant.

We suppose it known, not only that each of u and v has an amplitude and a unique sn, cn and dn, but that every θv, with θ a proper fraction, has. We are not told that $u+v$ or every $u+\theta v$ has. But if, on the as yet unproved assumption that these have, we can find values of their functions am, sn, cn, dn, and can then establish that these values do actually satisfy the defining differential equations, we shall be assured that the assumption is correct. Moreover, if the found values for sn, cn, dn are obtained as rational expressions in the known unique values for a limited range of arguments, the one-formedness, as well as the existence over the extended range, will have been proved.

On our assumption we can let our u, v in $u+v=$constant change continuously till they become $u+v$, 0. Let x', y' be the ultimate corresponding values of x, y. Then

$$x'+y'=x+y,$$

where, by (1),

$$\frac{\cos x'}{\mathrm{cn}(u+v)}=\frac{\sin x'}{\mathrm{sn}(u+v)}=\text{one of }\pm1=\frac{\cos y'}{1}=\frac{\sin y'}{0}.$$

These tell us that either

$$x'=\mathrm{am}(u+v),\qquad y'=0,$$

or

$$x'=\mathrm{am}(u+v)+\pi,\quad y'=\pi,$$

so that $x'+y'=$ either $\operatorname{am}(u+v)$ or $\operatorname{am}(u+v)+2\pi$,

between which we do not distinguish. Thus we have

$$\operatorname{am}(u+v)=x+y. \quad \text{.....(3)}$$

This is the fundamental addition equation. From it follow the ordinary formulae

$$\operatorname{cn}(u+v)=\cos x\cos y-\sin x\sin y=\frac{\operatorname{cn}u\operatorname{cn}v-\operatorname{sn}u\operatorname{sn}v\operatorname{dn}u\operatorname{dn}v}{1-k^2\operatorname{sn}^2u\operatorname{sn}^2v},$$

$$\operatorname{sn}(u+v)=\sin x\cos y+\cos x\sin y=\frac{\operatorname{sn}u\operatorname{cn}v\operatorname{dn}v+\operatorname{sn}v\operatorname{cn}u\operatorname{dn}u}{1-k^2\operatorname{sn}^2u\operatorname{sn}^2v},$$

$$\operatorname{cn}(u+v)\pm i\operatorname{sn}(u+v)=\frac{\cos x\pm i\sin x}{\cos y\mp i\sin y}=\frac{\operatorname{cn}u\pm i\operatorname{sn}u\operatorname{dn}v}{\operatorname{cn}v\mp i\operatorname{sn}v\operatorname{dn}u},$$

etc., etc.

As to $\operatorname{dn}(u+v)$, *i.e.* $\dfrac{d.\operatorname{am}(u+v)}{d(u+v)}=\dfrac{dx+dy}{du+dv}$, it is given by (2); or again, it is at once provided as $\dfrac{1}{i}\dfrac{\operatorname{cn}'(u+v)+i\operatorname{sn}'(u+v)}{\operatorname{cn}(u+v)+i\operatorname{sn}(u+v)}$.

The expressions for $\operatorname{sn}(u+\theta v)$, $\operatorname{cn}(u+\theta v)$, $\operatorname{dn}(u+\theta v)$ are in like manner, and on the same assumptions, given by putting θv for v in those just obtained for $\operatorname{sn}(u+v)$, $\operatorname{cn}(u+v)$, $\operatorname{dn}(u+v)$.

The verification that the values obtained do actually satisfy the differential equations requires only a little labour. Moreover, the expressions found are rational and free from ambiguity. Thus the assumptions made as to existence and one-formedness over the extended range of arguments are justified.

3. In order to apply this, we require knowledge of two finite ranges for u and v, over which one-formed values of sn, cn, dn are clearly specified.

First consider the real integral

$$u=\int_0^s\frac{ds}{\sqrt{(1-s^2)(1-k^2s^2)}},\quad\text{with } k^2<1,$$

the upper limit s being real and in the range -1 to $+1$.

As s grows continuously from 0 to 1, u grows continuously from 0 to

$$K=\int_0^1\frac{ds}{\sqrt{(1-s^2)(1-k^2s^2)}};$$

and as s diminishes from 0 to -1, u diminishes from 0 to $-K$.

Thus, corresponding to every real u from $-K$ to K, there is a real s. This s and the $+\sqrt{1-s^2}$, $+\sqrt{1-k^2s^2}$ which go with it are a definite sn, cn, dn of the u in question, the verification that they satisfy the differential equations being immediate.

Again consider the pure imaginary integral

$$iv=\int_0^{it}\frac{d(it)}{\sqrt{(1+t^2)(1+k^2t^2)}},\quad t \text{ real.}$$

As t grows continuously from 0 to ∞, v grows continuously from 0 to

$$K'=\int_0^\infty\frac{dt}{\sqrt{(1+t^2)(1+k^2t^2)}}\left(=\int_0^{\frac{1}{2}\pi}\frac{dT}{\sqrt{1-k'^2\sin^2T}},\ k'^2=1-k^2,\ \text{by } t=\tan T\right),$$

and as t diminishes from 0 to $-\infty$, v diminishes from 0 to $-K'$.

Thus, corresponding to every pure imaginary iv from $-iK'$ to iK', there is a pure imaginary it. This it and the $+\sqrt{1+t^2}$, $+\sqrt{1+k^2t^2}$ which go with it

are a definite sn, cn, dn of the iv in question; for, again, they satisfy the differential equations.

The addition equations of § 2 now give us without ambiguity values of sn, cn, dn for any $u+iv$, in which u has a real value between $-K$ and K, and v a real value between $-K'$ and K'.

Consequently we have quite definitely values of sn, cn, dn for all arguments $u+iv$ marked by points (u, v) in the finite rectangle enclosed by the lines $u=\pm K$, $v=\pm K'$ in the plane of rectangular axes of u and v.

4. Now, in the addition formulae of § 2, take for u an extreme $u+iv$ on the boundary of this rectangle, and for v any θ times the same $u+iv$, with $0<\theta\leqq 1$. The formulae give consistently and uniquely the sn, cn, dn of $(1+\theta)(u+iv)$; so that we now know uniquely the functions for all arguments whose marking points cover the doubled rectangle $(\pm 2K, \pm 2K')$. A second doubling, effected in like manner, extends knowledge to all arguments marked in a rectangle $(\pm 4K, \pm 4K')$; and by a succession of doublings we can extend it so as to apply to any finite $u'+iv'$ whatever.

Thus the existence and the one-formedness of functions $\mathrm{sn}(u'+iv')$, $\mathrm{cn}(u'+iv')$, $\mathrm{dn}(u'+iv')$ for all arguments $u'+iv'$ which we may have to contemplate are established. E. B. ELLIOTT.

MATHEMATICAL NOTES.

413. [L¹. 4.] *On some alternative forms of the equation of two tangents to a conic from an external point.*

Let two tangents TQ, TQ' be drawn from the point $T(hk)$ to the circle $x^2+y^2=a^2$, whose centre is at the origin O. Then taking any point $P(xy)$ upon either TQ or TQ' and drawing PL perpendicular to OT, we have

$$2\Delta OPT = PL \,.\, OT \quad \text{and also} \quad = PT \,.\, OQ.$$

Thus the equation of the tangents from T is

$$(kx-hy)^2 = a^2\{(x-h)^2+(y-k)^2\}.$$

This result (which has already appeared elsewhere) naturally seemed to lead to an enquiry whether the general form of the equation of the tangents from (x_1y_1) to a given conic which is written

$$(\text{Curve}) \,.\, (\text{Curve})_1 = (\text{Tangent})^2$$

may in some cases be further simplified as it is obvious that certain terms are the same on both sides. This proved upon experiment to be the case, the reason for which is explained by the following analysis.

If X, Y are current coordinates, the equation of the line joining two points (xy) and (hk) is $(X-x)(Y-k)=(X-h)(Y-y)$

or $$X(y-k)-Y(x-h)+(kx-hy)=0.$$

Now if $\phi(\lambda\mu\nu)=0$ is the tangential equation of any conic (or curve of a higher degree), by which is understood the relation subsisting between λ, μ, ν if $\lambda X+\mu Y+\nu=0$ is to be a tangent,—then the substitution of $y-k$, $-(x-h)$, $(kx-hy)$ for λ, μ, ν respectively will give the equation of the two (or more) tangents which can be drawn from (hk).

For instance $\lambda X+\mu Y=\nu$ touches the circle $X^2+Y^2=a^2$ if $a^2(\lambda^2+\mu^2)=\nu^2$; hence the equation of the tangents from (hk) is

$$(kx-hy)^2 = a^2\{(x-h)^2+(y-k)^2\},$$

as already obtained geometrically.

Similar equations may now be given:

Ellipse $x^2/a^2+y^2/b^2=1$	$(kx-hy)^2=b^2(x-h)^2+a^2(y-k)^2$.
Parabola $y^2=4ax$	$(y-k)(kx-hy)=a(x-h)^2$.
R.H. $xy=c^2$	$(kx-hy)^2=-4c^2(x-h)(y-k)$.

To go a step further, when the tangent to a curve involves a single parameter without employing the tangential equation at all we may at once identify the equation of the tangent with the standard form

$$X(y-k)-Y(x-h)+(kx-hy)=0$$

and eliminate the parameter.

The normal at ϕ to the ellipse is

$$aX\sec\phi-bY\operatorname{cosec}\phi=c^2,$$

and $$(y-k):(x-h):(kx-hy)=a\sec\phi:b\operatorname{cosec}\phi:-c^2.$$

Eliminating ϕ, we have

$$a^2/(y-k)^2+b^2/(x^2-h)^2=c^4/(kx-hy)^2$$

as the equation of the four normals which can be drawn from the point (hk) to the ellipse.

The corresponding equation for the parabola is

$$(x-h)^2\{(kx-hy)+2a(y-k)\}+a(y-k)^3=0.$$

The general equation of the tangents from (ξ, η) to the conic is found to be

$$\{(x-\xi)(C\eta-F)-(y-\eta)(C\xi-G)\}^2$$
$$=(-\Delta)\{a(x-\xi)^2+2h(x-\xi)(y-\eta)+b(y-\eta)^2\},$$

but it does not appear to have any special advantages.

As a further example, the straight line $lx+my=1$ will touch the parabola $\sqrt{x/a}+\sqrt{y/b}=1$ (axes oblique) if $1/al+1/bm=1$. Hence the tangents from (hk) have for their equation

$$-1/a(y-k)+1/b(x-h)=1/(kx-hy).$$

Parallels through the origin are $-1/ay+1/bx=1(kx-hy)$, and these will be at right angles if $h(a+b\cos\omega)+k(b+a\cos\omega)=ab\cos\omega$, giving the equation of the directrix.

The tangents from (hk) to the curve $(x/a)^m+(y/b)^m=1$ are represented by $\{b(x-h)\}^p+\{-a(y-k)\}^p=(kx-hy)^p$, where $p=m/(m-1)$.

The coordinates of any point on the cissoid $x(x^2+y^2)=ay^2$ may be taken as $am^2/(1+m^2)$ and $am^3/(1+m^2)$; the tangents from (hk) are represented by

$$\{(kx-hy)+a(y-k)\}^3=-2a^2(x-h)^2(kx-hy).$$

R. F. DAVIS.

414. [Q. 1. a.] *Non-Euclidean Geometry.*

I would like to lay before readers of the *Mathematical Gazette* a simple point supplementary to a paper which the editor so kindly printed for me last year. It is this. I have abandoned *Non-Euclidean Geometry* altogether, as one who trusts to direct intuition rather than logical refinement. My reason for rejecting the *hyperbolic* hypothesis was given in my paper, viz. the incontrovertibility, by simple geometrical procedure, of Bertrand's proof of the parallel-axiom. I expressed a hope that something similar would be found to dispose finally of the *elliptic* hypothesis (re-entrant straight lines, space of finite extent). For a long time I have felt a serious objection; but it is a physical and dynamical difficulty arising out of the Law of Gravitation. If l is the length of the entire straight line, the area of a sphere of radius r is $4\pi k^2\sin^2\frac{r}{k}$, where k is the finite space-constant, *i.e.* since $l=\pi k$,

$$\frac{4l^2}{\pi}\sin^2\frac{\pi}{l}.$$

Hence the only proper assumption for the law of gravitation would seem to be $\mu\operatorname{cosec}^2\frac{\pi r}{l}$, just as in the hyperbolic hypothesis one would assume $\mu\operatorname{cosech}^2\frac{\pi r}{l}$. I have worked out planetary motion for a force $\mu\operatorname{cosec}^2\frac{\pi r}{l}$;

but the work is all wrong, because if S is the sun and E the earth, then E is acted on by *two equal opposite forces*,

$$\mu \operatorname{cosec}^2 \frac{\pi r}{l} \quad \text{and} \quad \mu \operatorname{cosec}^2 \frac{\pi(l-r)}{l},$$

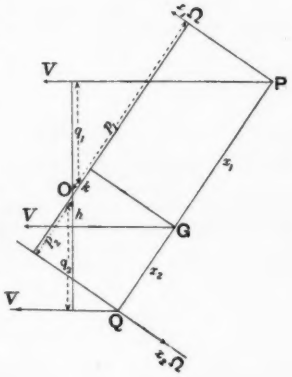
on this hypothesis. And so the elliptic hypothesis seems to be no use at all, for I cannot see how to find a reasonable form for the law of gravitation.

W. B. FRANKLAND.

415. [R. 2. d, 3.] *Angular Momentum of a body in two dimensions.*

The body may be supposed divided up into pairs of particles, the C.G. of each pair being G, the C.G. of the body. Let P and Q be one such pair of particles, m_1 and m_2 their masses. Let V be the velocity of G and Ω the angular velocity of the body about G.

Let O be any point in the plane of the body, p_1 and p_2 its distances from lines through P and Q $\perp^r$ to PQ, q_1 and q_2 its distances from lines through P and Q $\parallel$ to V, h its distance from V, and k its distance from the line through G $\perp^r$ to PQ.



Then, if

$$GP = x_1, \qquad GQ = x_2,$$

we have $\quad p_1 = x_1 + k, \quad p_2 = x_2 - k. \quad \ldots\ldots\text{(i)}$

Now the velocity of P is compounded of a velocity $x_1\Omega$ $\perp^r$ to GP, and a velocity equal and parallel to V, and similarly for Q.

Hence the angular momentum of the pair of particles about O is

$$\begin{aligned} &m_1 x_1 \Omega \,.\, p_1 + m_1 V \,.\, q_1 + m_2 x_2 \Omega \,.\, p_2 - m_2 V \,.\, q_2 \\ &= (m_1 x_1 p_1 + m_2 x_2 p_2)\Omega + (m_1 q_1 - m_2 q_2) V \\ &= (m_1 x_1^2 + m_2 x_2^2)\Omega + (m_1 q_1 - m_2 q_2) V, \end{aligned}$$

using (i) and remembering that $m_1 x_1 = m_2 x_2$.

Hence the angular momentum of the whole body is

$$[\Sigma(m_1 x_1^2 + m_2 x_2^2)]\Omega + [\Sigma(m_1 q_1 - m_2 q_2)] V. \quad \ldots\ldots\text{(ii)}$$

Now $\Sigma(m_1 x_1^2 + m_2 x_2^2)$ is the moment of inertia of the body about O (say, I). Also, since G is the C.G., we have, by the ordinary formula,

$$h = \frac{\Sigma(m_1 q_1 - m_2 q_2)}{\Sigma(m_1 + m_2)},$$

and $\Sigma(m_1 + m_2) = M$, the total mass, so that

$$\Sigma(m_1 q_1 - m_2 q_2) = Mh.$$

Hence, finally, from (ii), the whole angular momentum is

$$I\Omega + MVh,$$

i.e. the angular momentum due to rotation, together with that due to the motion of the C.G.

This method was suggested by an article on p. 364 of *A First Dynamics* (Jackson and Roberts), where it is applied to the simpler question of the kinetic energy of the body.

E. M. RADFORD.

416. [K¹. 20. e.] *Formulae for the area and half-angles of a triangle ABC obtained by means of the equivalent isosceles triangle ADE, viz. the isosceles triangle which has the same area and the same angle A as the given triangle.*

Also other formulae connected with the same diagram.

Data: $BC=a$, $AC=b=AC'$, $AB=c$, $AD=AE=\sqrt{bc}$.

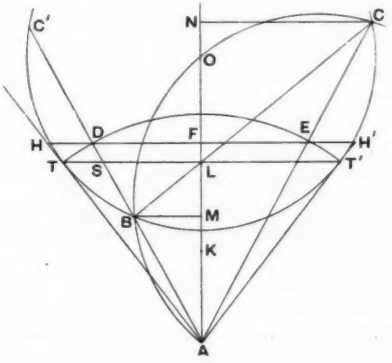
(1) $DE^2=AD^2+AE^2-2AD.AE\cos A$

$=2bc-2bc\cos A=2bc-(b^2+c^2-a^2)=4(s-b)(s-c)$;

$\therefore\ DF=FE=\sqrt{(s-b)(s-c)}$.

(2) $AF^2=AD^2-DF^2=bc-(s-b)(s-c)=s(b+c-s)=s(s-a)$;

$\therefore\ AF=\sqrt{s(s-a)}$.



Notes on diagram. AT, AT' touch the circle CBC' at T, T'; consequently H, H' lie slightly outside this circle.

Also K is so placed that $DK=FH=\frac{1}{2}BC$, and $FK=\frac{1}{2}BC'$.

Also O is the centre of the circle CBC'.

Hence area of triangle $=\sqrt{s(s-a)(s-b)(s-c)}$.

$$\sin\tfrac{1}{2}A=\sqrt{\frac{(s-b)(s-c)}{bc}};\qquad \cos\tfrac{1}{2}A=\sqrt{\frac{s(s-a)}{bc}};$$

$$\tan\tfrac{1}{2}A=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}};$$

and $$s(s-a)+(s-b)(s-c)=bc.$$

Connected with the diagram, but not using the isosceles triangle, is a method for finding $\tan\frac{1}{2}(B-C)$ given me by Dr. Heath, Vice Principal of Birmingham University, who has kindly permitted me to print it here.

If BM, CN be drawn perpendicular to AF, $ABM=\frac{1}{2}(B+C)$; consequently $LBM=\frac{1}{2}(B-C)$;

$$\therefore\ \tan\tfrac{1}{2}(B-C)=\frac{MN}{CN+MB}=\frac{(b-c)\cos\frac{1}{2}A}{(b+c)\sin\frac{1}{2}A}.$$

Constructions connected with the auxiliary angle methods of calculating BC.

Writing the formula $a^2=b^2+c^2-2bc\cos A$

in the form $$a^2=(b+c)^2\left(1-\frac{4bc\cos^2\frac{A}{2}}{(b+c)^2}\right),$$

so that $$a=(b+c)\sin\phi,\ \text{if}\ \cos\phi=\frac{\sqrt{bc}\,.\cos\frac{1}{2}A}{\frac{1}{2}(b+c)},$$

then, since $AF=\sqrt{bc}\cos\frac{1}{2}A$, we see that if FD is produced to H, so that $FH=\frac{1}{2}a$, then $AH=\frac{1}{2}(b+c)$, and the angle $HAF=\phi$.

The line AH is of special interest. For let the bisector of the angle A meet BC in L and the circumcircle of ABC in O, then the circle with centre O, radius OC, will pass through B and C'. Draw LT perpendicular to AL, to cut this circle in T, and join AT.

Then $$bc=AL^2+BL\,.\,LC\quad\text{(Euc. VI. B.)}$$
$$=AL^2+LT^2=AT^2;$$

$\therefore$ AT touches the circle CBC' at T, and its length $=\sqrt{bc}=AD=AE$.

(This construction is incidentally a good one for constructing the triangle ADE.)

Also, since $$2\triangle ABL+2\triangle ALC=2\triangle ABC,$$

we have $$AL(b+c)\sin\tfrac{1}{2}A=bc\sin A;$$

$$\therefore\ AL=\frac{2bc\cos\frac{1}{2}A}{b+c};^*\quad\therefore\ \frac{AL}{AT}=\cos\phi.$$

Hence, if the tangent AT be produced, it will cut DF produced in H, such that $AH=AF\sec\phi=\frac{1}{2}(b+c)$, and consequently $FH=\frac{1}{2}a$, as before.

I have not seen a geometrical construction of this auxiliary angle ϕ before, and its simplicity, being merely the angle between the tangent from A to the circle BCC' and the bisector AL, is very enchanting.

It is easy to give a construction for the other auxiliary angle ϕ' given by the formula $$a^2=(b-c)^2+4bc\sin^2\tfrac{1}{2}A,$$

whereby $$a=(b-c)\sec\phi',\ \text{if}\ \tan\phi'=\frac{\sqrt{bc}\sin\frac{1}{2}A}{\frac{1}{2}(b-c)}=\frac{DF}{\frac{1}{2}(b-c)},$$

by taking a point K on the bisector, such that $FK=\frac{1}{2}(b-c)$, and consequently $DK=\frac{1}{2}a$; then $DKF=\phi'$.

This, however, is rather artificial, and therefore of no special interest, whereas ϕ came naturally during the process of finding the lengths of AD, AE; moreover, the circle concerned is an important one, as it cuts AL in the in-centre and an ex-centre of ABC. A. Lodge.

417. [C. 2. a.] *Geometrical Integration of* $\sec\theta\,d\theta$.

Let Q be a point in the arc AQB of a quadrant of a circle AOB of radius a, $\angle AOQ=\theta$, Q' a pt. near to Q.

Draw QN perpr. to OA, and let BQ, BQ', QQ' meet OA produced in P, P', T.

Then $$PQ\,.\,PB=PO^2-OQ^2=PB^2-2OQ^2;$$
$$\therefore\ BP\,.\,BQ=2OQ^2=BP'\,.\,BQ';$$
$$\therefore\ P,\ P',\ Q',\ Q\ \text{are concyclic.}$$

$\therefore$ (i) $\triangle$s BQQ', BPP' are similar, and (ii) $\angle TQP=\angle TP'Q$.

From (i), $$\frac{PP'}{QQ'}=\frac{BP}{BQ},$$

$$\frac{1}{a}\cdot\frac{d(OP)}{d\theta}=\frac{BP}{BQ}=\frac{OP}{ON};$$

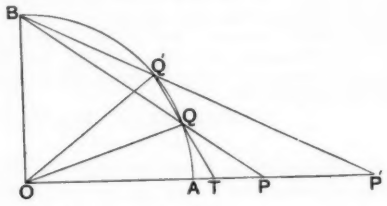
*It might just be noticed that AS, AD (or AT), and AH are the harmonic, geometric, and arithmetic means between b, c.

$$\therefore \frac{d(OP)}{OP} = \frac{d\theta}{\cos\theta};$$

$$\therefore \int \frac{d\theta}{\cos\theta} = \log OP$$

$$= \log(\sec\theta + \tan\theta),$$

for in limit $TP = TQ = a\tan\theta$; $OT = a\sec\theta$.



Also, since $\angle ABQ = \frac{\theta}{2}$, $\angle OBQ = \frac{\pi}{4} + \frac{\theta}{2}$;

$$\therefore \sec\theta + \tan\theta = \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right).$$

E. M. LANGLEY.

418. [F. 8. d. β, γ.] Notes 373, 381, 397: vol. vi. pp. 338, 379; vol. vii. p. 109.

It is easy to see that the right-hand members of the relations (i)-(v) and of the last on Note 373 are the differences of two squares.

Taking into account also the decomposition of (i) into four squares and of (v) into six squares as in 397, it is readily seen that the following solutions of the indeterminate equation $u^2 = v^2 + w^2 + t^2$ are at once obtained:

(i) $u = x^6 + 8x^4y^2 + 8x^2y^4 + 2y^6$; $v = x^6$;
$w = 2y^4(2x^2 + y^2)$; $t = 2xy(2x^4 + 5x^2y^2 + 2y^4)$.

(ii) $u = x^6 + 2y^6$; $v = x^2(x^4 - 8x^2y^4 + 8y^4)$;
$w = 2y^4(2x^2 - y^2)$; $t = 2xy(2x^4 - 5x^2y^2 + 2y^4)$.

(iii) $u = 8x^6 + 72x^4y^2 + 189x^2y^4 + 81y^6$; $v = 4x^4(2x^2 + 9y^2)$;
$w = 27y^4(x^2 + 3y^2)$; $t = 6xy(4x^4 + 24x^2y^2 + 27y^4)$.

(iv) $u = 2x^6 + 174x^4y^2 + 36x^2y^4 + y^6$; $v = 2x^4(x^2 - 75y^2)$;
$w = y^4(18x^2 + y^2)$; $t = 2xy(18x^4 + 55x^2y^2 + 3y^4)$.

(v) $u = x^6 + 16x^4y^2 + 72x^2y^4 + 162y^6$; $v = x^4(x^2 + 8y^2)$;
$w = 18y^4(2x^2 + 9y^2)$; $t = 2xy(2x^4 + 21x^2y^2 + 54y^4)$.

The decomposition into four squares in Note 397 gives the solution of the following equation, $u^2 = v^2 + w^2 + t^2 + p^2 + q^2$:

$$u = x^6 + 8x^4y^2 + 8x^2y^4 + 2y^6;\ v = 2xy(2x^4 + 5x^2y^2 + 2y^4);$$
$$w = x^6;\ t = 4x^3y^4;\ p = 4xy^5;\ q = 2y^6.$$

The decomposition into six squares in the same note gives the solution of $u^2 = v^2 + w^2 + t^2 + p^2 + q^2 + r^2 + s^2$:

$$u = x^6 + 16x^4y^2 + 72x^2y^4 + 162y^6;\ v = 2xy(2x^4 + 21x^2y^2 + 54y^4);$$
$$w = x^6;\ t = 4x^5y;\ p = 8x^4y^2;\ q = 36x^2y^4;\ r = 108xy^5;\ s = 162y^6.$$

E. N. BARISIEN.

ANSWERS TO QUERIES.

[85, p. 246, vol. vii.] I had it second-hand from the director of an astronomical observatory, that he insisted on his computers writing 0·11 and not ·11, as there was no chance of the former being misread as 11. C. W. ADAMS.

[85, p. 246, vol. vii.] I generally put 0·125 for one-eighth expressed as a decimal, because if the point slips out in printing it is still reasonably obvious what is meant. R. W. K. E.

[87, p. 247, vol. vii.] If x, y, z are the tripolar coordinates of $P(x=PA^2$, etc.), and Q be the point whose trilinear coordinates are (α, β, γ), then

$$2\Delta \cdot PQ^2 = \Sigma a\alpha x - 2R\Sigma a\beta\gamma$$

(*Math. Gaz.*, p. 186, vol. ii.),

from which the result required follows immediately. G. N. BATES.

REVIEWS.

Differential and Integral Calculus. By LORRAIN S. HULBURT. Pp. xviii, 481. 1912. (Longmans, Green & Co.)

This book has many good points. It is readable and fairly accurate, and the examples are simple and interesting. The geometrical parts are the best, but even in the analytical parts a good deal is done correctly which is bungled hopelessly in many books with wide circulations: I may instance the treatment of differentials and the differentiation of x^n. The author does not pretend to be rigorous in his treatment of fundamentals, and the compromise which he attempts to set up between rigour and simplicity is often a very reasonable one. Sometimes he is less successful; and I append a few criticisms of particular passages, which might be useful if the book should reach a second edition.

P. 5. $3-(2-1)=7-(3+2)$ is certainly not an 'identity,' for it contains no variables. The author's example contradicts his own definition.

Pp. 10 *et seq.* The author seems to suggest that a discontinuity of a function is necessarily accompanied by a failure in its definition.

P. 68. The 'definition' of an 'increasing function' is not a definition at all, but a mere tautology.

Pp. 88 *et seq.* The treatment of the exponential limit is bad. The author *does* profess to prove that

$$\lim\left(1+\frac{1}{n}\right)^n = e,$$

when n is restricted to be a positive integer, but his proof is fallacious. He then tacitly assumes that his proof applies to the case in which $n \to -\infty$. He recognises that he has not proved everything, but not that he has proved nothing.

Pp. 240 *et seq.* All the discussion of the definite integral is also bad. It would have been much better to give no proof at all. The proof of the existence of a definite integral is substantially simply a proof that a certain type of area exists. If the latter proposition, which involves all the difficulties of the former, is to be assumed, nothing can be gained by not assuming the former as well. The most that can be done profitably is to give the obvious geometrical reasons for supposing the two problems to be identical.

Pp. 388 *et seq.* To define $e^{i\theta}$ as meaning $\cos\theta + i\sin\theta$, and then to 'deduce' De Moivre's theorem by assuming that $(e^{i\theta})^n = e^{ni\theta}$, is palpably absurd.

Pp. 410 *et seq.* The use of the term 'consecutive' in the treatment of envelopes is unfortunate.

I may add that I cannot regard an elementary mathematical book as a very suitable place for instruction in the rudiments of French, and that in any case I have grave doubts whether "Kō'sheé" is a very accurate phonetic rendering of the name of the great mathematician.

Leçons de Mathématiques Générales. Par L. Zoretti. Pp. xvi+753. 1914. (Gauthier-Villars.)

This is a very remarkable book, to which I should like to call the attention of schoolmasters and of all whose business it is to teach mathematics to intelligent pupils who are not mathematical specialists.

I should hardly have thought it possible to cover so much ground in a single volume. M. Zoretti assumes no knowledge in his reader beyond that of a little geometry, algebra, and trigonometry; and he carries him far beyond the limits of Part I. of the Mathematical Tripos. There are chapters, for example, on elliptic functions, Fourier's Series, partial differential equations, vectors in space, and elementary differential geometry of two and three dimensions. And in spite of the extraordinary variety of the subjects with which he deals, M. Zoretti always seems to be treating them at his leisure and to have plenty of space to spare.

Naturally M. Zoretti does not profess to give a full and rigorous treatment of fundamentals. He does not prove, for example, that a series of positive terms must converge or diverge to infinity, or establish the existence of implicit functions or of the definite integral. But he is always perfectly clear and consistent. "Ma méthode est ... admettre franchement tout ce qui présente des difficultés sérieuses, ou simplement de trop longs calculs; donner, quand c'est possible sans longueurs, soit la démonstration, soit au moins ses grandes lignes."

Herein lies the great difference between such a book as this and the majority of the English books on "Practical Mathematics" which appear in such profusion nowadays. M. Zoretti is "practical" enough. He has a most admirable chapter on methods of numerical calculation, calculating machines, and so forth, a chapter which, ignorant as I am in such matters, I read with the greatest interest: and in general he has the possibility of practical application always before him. But with him to be "practical" does not mean, as too often with English writers, merely to be ignorant of all that makes mathematics a science worth learning. For M. Zoretti is a real mathematician who has done distinguished work in the theory of functions, and knows perfectly well where the real difficulties of analysis lie. G. H. Hardy.

Cours de Mécanique professé à l'Ecole Polytechnique. Par Leon Lecornu. Tome 1. Pp. 536. 1914. (Paris: Gauthier-Villars.)

The École Polytechnique, the greatest of military schools, was founded by a personage, of some note in his day as a practical soldier, who held an opinion as to the value of mathematics in military education which is not much in fashion in England. The tradition of the school has been faithful to the doctrine of its founder, and a severe training in mathematics has always been exacted from the competitors for entrance, and continued throughout their career at the school. M. Lecornu, in his preface, quotes some interesting remarks by General Langlois to the effect that the officers who pass out after only one year of training are definitely inferior to their comrades who have completed the full course, whenever a question arises involving a thorough investigation of some tactical problem, and he adds: "The method of work indispensable for every man of action demands absolutely the study, ending in mastery, of some branch of science; a study which exercises the intellect and confers that faculty of logical deduction indispensable to the commander."

In this country great virtue is attached, and rightly, to the faculty of decision, but it is sometimes forgotten that decision only becomes a virtue difficult of practice and worthy of honour when it is exercised by a person who knows and feels how much there is to be said on both sides of the question. The remarks of General Langlois offer an impressive caution which the practical man of to-day will do well to bear in mind.

Coming more definitely to the subject-matter of the book, it is a book on theoretical mechanics for the education of men who will ultimately have to deal with grave practical questions of applied mathematics. Consequently the theory developed is illustrated, but not replaced, by references to concrete applications.

The present volume contains a brilliant and lucid sketch of kinematics, including the kinematics of machinery, statics, and the dynamics of a particle.

At first sight, the absence of numerical examples appears a defect, but it must be remembered that it is an essential feature of the traditional French scheme of work that the lecture of the professor shall be supplemented by the "répétiteur" who deals with the direct and simple applications of the theories expounded.

In one respect, it may be suggested with all deference, this division of labour is not entirely satisfactory.

In dealing, for instance, with the damped movement of a spring the author adopts as his differential equation

$$m\frac{d^2x}{dt^2}+2mh\frac{dx}{dt}+m\omega^2x=0,$$

where $m\omega^2x$ is the restoring force of the spring, and $2mh\dfrac{dx}{dt}$ is the resistance, proportional to the velocity.

The constants ω^2 and h have no simple physical meaning, and the dimensions of the various terms in the equation or in its solution are not obvious. If we write $W\dfrac{x}{a}\left(\text{or } mg\dfrac{x}{a}\right)$ for the restoring force, and $W\dfrac{v}{v_0}$ for the resistance, a and v_0 have direct physical interpretations; namely, the elongation of the spring when the restoring force is mg, and the velocity when the resistance is mg. The equation becomes

$$\frac{d^2x}{dt^2}+\frac{g}{v_0}\frac{dx}{dt}+g\frac{x}{a}=0,$$

and any result of integration, for velocity, time, or distance, protrudes the evidence of its correctness from the point of view of dimensions.

The appearance of the second volume will be awaited with the interest which every production from so famous an institution must command.

C. S. JACKSON.

A History of Japanese Mathematics. By DAVID EUGENE SMITH and YOSHIO MIKAMI. Pp. vii+288. Cloth, 3 dollars or 12s. net. 1914. (Chicago and London: The Open Court Publishing Co.)

Until very recently (from 1900 on) the mathematics of China and Japan has been closed to all European and American students. The aim followed by Prof. Smith and Mr. Mikami has been to give a brief survey of the leading features in the development of the "wasan" or native Japanese mathematics as distinguished from the "yosan" or European mathematics. The European literature, the general planning of the work, and the final writing of the text is chiefly the work of Prof. Smith; while Mr. Mikami is responsible for furnishing the Japanese material, the initial translations, search through the library of the Academy of Sciences of Tokio (where T. Endo, the greatest of the historians of Japanese mathematics, is librarian), and a further examination of the large amount of native secondary material. The fourteen chapters are concerned respectively with: The earliest period; the second period; the development of the "soroban" (the Japanese abacus); the "sangi" (roots used in computing, and as numerical coefficients in equations) applied to algebra; the third period (1600 to the beginning of Seki's influence, about 1675); Seki Kowa (1642-1708); Seki's contemporaries and possible Western influences; the "yenri" or circle principle; the eighteenth century; Ajima Chokuyen (1739-1798); the opening of the nineteenth century; Wada Nei (1787-1840); the close of the old "wasan"; the introduction of occidental mathematics. The first period extends from the most remote times to A.D. 552, and was influenced only indirectly, if at all, by Chinese mathematics. The second period (552-1600) was characterised by the influx of Chinese learning, by some resulting native development, and by a period of stagnation: "... a period of a thousand years, with not a single book of any merit, and without advancing the science of mathematics a single pace. Europe was backward enough, but Japan was worse. ..." (p. 17). The third period was

less than a century in duration, extending from about 1600 to the beginning of Seki's influence (about 1675). This may be called the Renaissance period of Japanese mathematics, since it saw a new and vigorous importation of Chinese science, the revival of native interest through the efforts of the immediate predecessors of Seki, and some slight introduction of European learning through the early Dutch traders and through the Jesuits (p. 1).

The attitude of the "samurai" and the guild of learning towards the plebeian "soroban" (p. 47) reminds one strongly of the attitude of the ancient Greek philosophers to the arithmetic of commerce. The science of arithmetic developed very early in the far East; thus the distinction between positive and negative is very old (p. 48), but imaginaries were never considered in pure Japanese mathematics (p. 209). Chinese algebra, we are told, took a great step forward in the thirteenth century (pp. 48-50). In the solution of numerical equations China and Japan were actually first of all nations. Horner's method had long been known in the East (pp. 213, 279). As regards geometry in Japan, conic sections were often investigated (pp. 68, 206, 279).

The name of Seki has long been associated with the "yenri" or "circle principle," a form of the calculus that was possibly, though by no means certainly, invented by him (pp. 94, 95, 127, 143, 151 sqq.). Seki also made improvements in algebra, and actually antedated Leibniz in the discovery of determinants (pp. 95, 121-127). Still, apart from the achievements just mentioned, Seki's "reputation has no basis in any great field of mathematics. That he was a wonderful teacher there can be no doubt; that he did a great deal to awaken Japan to realise her power in learning no one will question; that he was ingenious in improving mathematical devices is evident in everything he attempted; but that he was a great mathematician, the discoverer of any epoch-making theory, a genius of the highest order, there is not the slightest evidence. He may be compared with Christian Wolf rather than Leibniz, and with Barrow rather than Newton" (p. 127).

Of the "yenri" itself the authors say (p. 150) that it "is simply the interesting, ingenious, rather obscure method of deriving a formula capable of being applied in several ways, but that it is no more comparable to the European calculus, even as it existed in the time of Seki, than is Archimedes's method of squaring the parabola, while the method is stated with none of the lucidity of the great Syracusan." A series which is the essence of the "yenri" is possibly of Western origin (p. 155); and the "yenri" at first was little more than a rather primitive method of using infinite series in the measurement of the simplest curvilinear figures and the sphere. Later, in the hands of Ajima Chokuyen and Wada Nei, it gradually unfolded into something more elaborate, but it never became a great method and remained always a set of ingenious devices (pp. 162, 201, 204-205, 220, 221, 230). On pp. 193-194 the authors "characterise the eighteenth century as one of problem-solving, of the extension of a rather ill-defined application of infinite series to the mensuration of the circle, of some slight improvement in the various processes, of the rather arrogant supremacy of the Seki school, and of a bitter feud between the independents and the conservatives in the teaching of mathematics. All this is a fair characterisation of most of the latter half of the century. There was, however, one redeeming feature, and this is found in the work of Ajima Chokuyen."

The opening years of the nineteenth century were characterised by a greater infiltration of Western learning, by some improvement in the algebra of the Seki school, and by the initial steps in pure geometry. There was one name of distinct importance in the early years of the century—the name of Wada Nei (p. 219). There was then some activity in the field of the "yenri" until the opening of Japan to foreign commerce (1868) and the consequent close of the "wasan" (pp. 230, 255).

Possibly there was a slight connection between the mathematics of Japan and Europe in the seventeenth century. The possibility of such a connection, chiefly through Dutch traders and Jesuit missionaries, increased as time went on, and in the nineteenth century the mathematics of the West finally usurped the place of the "wasan" (pp. 254-256, 259, 263). A mention of an independent Japanese version of the nebular hypothesis of Kant and Laplace (p. 263),

and of the fact that the differential and integral calculus, in its Western form, entered Japan through a Chinese version (1859) of Loomis's text-book of 1850 (p. 274) seem of special interest.

"The mathematics of Japan," conclude the authors on pp. 279-280, "was like her art, exquisite rather than grand. She never developed a great theory that in any way compares with the calculus as it existed when Cauchy, for example, had finished with it. Japan had had hardly a century of real opportunity in mathematics when Seki entered the field. From the standard of opportunity Japan did remarkable work, from the standpoint of mathematical discovery this work was in every way inferior to that of the West. When, however, we come to execution it is like picking up a box of the real old red lacquer,—not the kind made for sale in our day. In execution the work was exquisite in a way wholly unknown in the West. For patience, for the everlasting taking of pains, for ingenuity in untangling minute knots and thousands of them, the problem-solving of the Japanese and the working out of some of the series in the 'yenri' have never been equalled."

On p. 189 we read of "what is called in English a 'tempest in a tea-pot.'" This is, so far as I see, the only place where some difficulty has been found by Mr. Mikami in acquaintance with English phrases. This point brings us to another. It is impossible not to compare this book with Mr. Mikami's *Development of Mathematics in China and Japan* (19 marks, Teubner) published last year. It is a curious coincidence that both this work and the work under review were printed in Germany, written in English, and revised by Americans. But while Dr. G. B. Halsted, who revised Mr. Mikami's former book, kept almost wholly the quaint idiom used by the author, Prof. Smith has almost entirely avoided it. I think that Prof. Smith is right. In a scientific work the difficulties that we or others may have about the language it is written in are irrelevant and confusing. Neither is there anything that deserves preservation as literature in a book of broken or chipped English. It is only perhaps from the point of view of not very exalted humour that Dr. Halsted seems justified. The work under review, though printed in Germany, is published in Chicago and London, and thus seems to bear evidence, even more than the former work, to the truly international character of science. Prof. Smith and Mr. Mikami seem to take rather a melancholy tone (p. 2) about the loss of national characteristics in the mathematics of Japan, and the passage from the "wasan" to international mathematics. This melancholy will not be shared by those who value a science more than those who helped to discover parts of it; but all will agree that Prof. Smith and Mr. Mikami have produced a book which is really of absorbing interest, whether we are more interested in the Japanese or in what they discovered and how, often independently of Western influences, they discovered it.

The Principle of Relativity in the Light of the Philosophy of Science. By PAUL CARUS. With an Appendix containing a Letter from the Rev. James Bradley on the Motion of the Fixed Stars, 1727. Pp. iii, 105. Cloth, 1 dollar, or 4s. net. 1914. (Chicago and London: The Open Court Publishing Co.)

This book consists of three articles reprinted from *The Monist* for 1912. Dr. Carus's view is that the theory of relativity is not a proposition whose truth can be found by experiment, but "a method, and the problem is simply how we can best deal with certain difficulties due to the relativity of all things" (p. 2). At the outset, Dr. Carus, conservative in spirit but eagerly receptive of new ideas, states his thesis: ". . . the relativity physicists are perfectly right; what they claim is really and truly a matter of course, and if they only would present their proposition without dressing up their theory in paradoxical statements, nobody would in the least hesitate to accept the new view. But as soon as this is done people will at the same time find out that the new view is not novel. Its importance has been greatly exaggerated, for the principle has been tacitly understood in the correct way by all preceding physicists who, at the time however, ignored, or better did not enter into, the problem, because they had other more pressing work on hand. Nor is it unlikely that they regarded this problem of relativity as a philo-

sophical question which, strictly speaking, had no place before the forum of physics" (p. 3). The relativity of time and space was recognised by Bradley, whose famous paper on the motion of the fixed stars is here reprinted from the *Philosophical Transactions* for 1727. The expositions of Comstock (pp. 12-15), Cohn, Franck, and others (pp. 22-34), Mach (pp. 34-39), Campbell (pp. 43-51), Magie (pp. 54-62), Michelson (pp. 66-70), and Huntington (pp. 79-80) are then dealt with.

The second postulate of the theory of relativity (pp. 12, 43) would seem not to have been anticipated by the older physicists, and perhaps Dr. Carus does not give, in consequence, quite enough credit to the modern physicists. Another ground of criticism is as follows. We agree that reality is a continuous flux, but we cannot agree that the mathematician, when working with the differential calculus, splits up a curve into infinitesimal straight lines, much as a kinematoscope splits up motion (p. 9). This is certainly true of the way in which many applied mathematicians work, but the calculus can be quite logically founded on *limiting* processes, in which any apparent need for helping our minds by analysing a continuum into a *finite* series of atoms (or static states, as the case may be) vanishes.

The greatest value of this book seems to me to lie in the great truth that there is no hard line separating "philosophy" and "science." Men of science, as Stallo has so admirably shown, continually and unconsciously play the parts of amateur and very ignorant philosophers, especially if they affect to despise philosophy. We daily hear of scientific Cartesians who have never read a page of Descartes, and who only deserve the name "Cartesians" because they have many of the defects of Descartes. Dr. Carus (cf. pp. 10, 15, 19-20, 40, 82) is one of those who try to convince us, and do convince the intelligent ones among us, that knowledge is one, that what we can get to know in philosophy—it is not so much as many philosophers have thought—is as scientific as that the earth revolves or that twice two are four; and that science is after all what Newton called it—"natural philosophy." Men of science often object to *à priori* as being a "metaphysical" term. It is a logical term; and to dispute Dr. Carus's position that the principle of relativity is *à priori* (cf. pp. 39, 75) is like disputing the truth of the Pythagorean theorem because all the lines that we can make are too thick.

The Mechanistic Principle and the Non-Mechanical. An Inquiry into Fundamentals, with Extracts from Representatives of either side. By PAUL CARUS. Pp. iv, 125. 1913. Cloth, 1 dollar, or 4s. net. (Chicago and London: The Open Court Publishing Co.)

"On the one hand," says Dr. Carus on p. 3, "we claim that all causation is mechanical in the strict sense of the word according to the mechanistic principle; every cause is a motion, every result is accomplished according to conditions and circumstances by changes of place, and all processes that take place are transformations. On the other hand, we accept the belief that not only men but even less highly organised creatures are purpose-endowed, and above all that the regularities of natural laws, the lawdom of nature, is so intrinsic as to constitute evolution in both the inanimate and the organised domains of existence. This order of the universe is its most inalienable feature, which shows itself in a definite direction of development, and in a growth with increasing purpose, a predetermined end or aim called *telos* in Greek, and though the world-order is not a man-made design, it may very well be compared to a plan, and is analogous to a premeditated purpose; it acts like one and may be represented as such." "Motions are mechanical, and the atoms of a brain move and cannot help moving in a definite way prescribed by the laws of motion, just as a stone falls to the ground and as the comets sweep through the heavens in perfect agreement with the laws of gravitation. However, this truth is not contradicted by the fact that every living being, and especially man, is endowed with purpose" (p. 18). "We will illustrate the situation by comparing the brain to a book. The spirit of the living brain consists in the meaning which the several feelings possess, and the spirit of a book is of the same kind. Meaning is an impalpable something; it is neither material nor mechanical, yet it is the main

portion of a living person and of a book. Meaning is, as it were, the stuff of which spirit consists. . . . Analyse a book, you cannot discover its meaning in the most minute products of the analysis in the chemist's crucible" (pp. 22-23). These seem to be the fundamental doctrines in a book which is clearly written and often happily illustrated. The criticisms of the procedure of purposivists are rendered more forcible by the author's sympathetic position.

The first thinker who ventured to pronounce that if causation prevails at all it must be mechanical was Lamettrie (pp. 7, 98-110); and the opinions of "Mark Twain" (pp. 54-97), Prof. W. B. Smith (pp. 111-115), and Dr. J. T. Bixby (pp. 116-125) are discussed. It must be mentioned that "Mark Twain's" philosophy is exposed very fully by actual quotations, and this is the only generally accessible place where it can be consulted. Dr. Carus deserves our thanks for preserving the story on pp. 67-70.

PHILIP E. B. JOURDAIN.

Practical Mathematics for Technical Students. Part I. T. S. USHERWOOD and C. J. A. TRIMBLE. Pp. 370. Price 3s. 6d. (Macmillan.)

That the average technical student is a difficult man to provide with a textbook is a matter beyond question: and the number of really successful attempts to write a book which he will favour, and at the same time one which will do him real and lasting good, is exceedingly small. This volume must, we believe, be placed on this small list.

The authors state in their preface that they have endeavoured consistently to avoid any form of treatment which might lead the student to regard the different aspects of mathematical work as essentially distinct subjects. This idea, carried out in the way the authors have done it, makes for vigour and interest. The whole work is essentially "live." Of course, the methods employed are not always strictly academic, but great judgment has been shown in the occasional use of rigorous treatment of fundamentals. I believe Prof. Perry is responsible for the maxim that "a good teacher will occasionally intersperse his teaching of practical mathematics with rigorous demonstration, *but only if his students take a delight in it.*" My experience has been that this introduction of rigorous treatment has another, a more important, side; it inculcates *confidence* in the truth of the rules and, as the student progresses, he inevitably acquires a certain reverence for the infallible nature of mathematics as an exact science. It may be that on this is founded later an appreciation of, if not a distinct liking for, the subject. This book would not have suffered if more of the explanatory matter had given place to fuller academic treatment.

The authors tacitly insist throughout that all letters stand for numbers and not concrete quantities, and so they overcome the great bugbear of every technical student. In decimals they use the method of rough checks for placing the decimal point, but I think are then illogical in setting the unit figure of the multiplier under a definite digit of the multiplicand; nor is there, as far as I can see, any discussion as to the accuracy of the last figure in $6{\cdot}222\times10^{-2}$ as the product of $24{\cdot}86\times4{\cdot}001\times0{\cdot}0006256$, although they have all the material for this collected as an introduction to the chapter. In my opinion, this is rather an important omission; for, if the original numbers are *numbers correct to four significant figures*, all that is known is that the product lies between $6{\cdot}220\times10^{-2}$ and $6{\cdot}229\times10^{-2}$, so that the last figure given may be a 2 or a 3, *and most probably a* 3. The student should be taught that the fourth figure is rarely reliable, but that on the other hand the error rarely exceeds one, in that figure. The same remark applies to four-figure logarithms, the rules of which are excellently explained from the idea of numbers in their "Standard Form." Graphs form another good section, but the models of tabular computation are not to be recommended; the introduction of the "working" into the tabular statement is horribly confusing to the student—and to an examiner. The Slide Rule section might well have been omitted or curtailed: the use of this "tool" should always be a matter for personal, and where possible, individual tuition. Rates receive very careful consideration, and the section on mensuration is

good and well illustrated with interesting examples of every-day occurrence in the out-of-class life of the student.

I believe I have found every fault possible (as is seen, not much) with this book, which, with sets of tables and specimen examination papers to complete it, is well worth a trial by those who are dissatisfied with the book they are using.

Test Papers in Elementary Algebra. By C. V. DURELL. Pp. 233. Price 3s. 6d. (Macmillan.)

This collection of 200 test-papers should prove of good service. Though of the usual school and examination type in general, some of the questions are distinctly original, and should give even the best boys food for thought. On the whole, they are very carefully graduated; nevertheless there are sometimes incongruities, such as setting a boy, who is supposed to be acquainted with functional notation and the remainder theorems, a question such as: "If $a+b=15$, $b+c=16$, $c+d=22$, find the value of $a+d$." But the whole key-note of the compilation is—"Don't forget the boy near the bottom of the class": very useful.

Exercises in Mathematics. By DAVID BEVERIDGE MAIR, M.A. Pp. 468. Price 3s. 6d. With answers and hints, 4s. 6d. 1914. (Macmillan.)

A very useful set of exercises on School Mathematics: many of the examples are from examination papers, but a considerable number, by no means the least valuable, appear to be original. There are in all 300 pages of these exercises; then follow nearly 50 pages of specimen papers set by the chief public examining bodies. The Answers, 140 pages in extent, contain hints towards solution in suitable cases; this should render the book of great service to private students and the "form-master" who does not profess to be a mathematician. Tables of weights and measures occupy the front and tables of logarithms the back fly-leaves.

The Groundwork of Arithmetic. By MARGARET PUNNETT, B.A. Pp. 234. Price 3s. 6d. 1914. (Longmans, Green.)

Exercises in the Groundwork of Arithmetic. Parts I., II., III., 4d., 6d., and 6d. respectively. 1914. (Longmans, Green.)

Intended as a five-years' course for children from 6 to 11 years of age, the text and the exercises to accompany it form a really inspiring work that every elementary teacher should have on his or her bookshelf: nor can I imagine a teacher of older children reading through the text without obtaining considerable benefit thereby. Personally, I started "looking through it" for the purposes of criticism and ended by reading every word.

I think, however, that in the last two sections, the point so carefully brought out in the first section, that 3 is only the *conventional symbol* for the number three, might have been pushed further and literal symbols have been introduced, starting first say with 1, ⌐, ⌐ (or Δ), □, etc., for the numbers 1, 2, 3, 4, etc., with the introduction of such formulae as $x=2y+3z$, $a=b\times c$, all letters standing for natural numbers.

The Teaching of Arithmetic. By DAVID EUGENE SMITH. Pp. 196. Price 4s. 6d. 1913. (Ginn.)

This is another interesting volume on the presentment of Arithmetic to the young pupil. It goes a little further than Miss Punnett's book, but with less detail. There is some allusion to the introduction of literal symbols, such as the use of x as a "shorthand" symbol in simple problems involving division or proportion, but that is all. Surely by the age of 9-10 every boy ought to have some sort of notion of such a formula as $l\times b$ for the measure of the area of a rectangle.

J. M. CHILD.

Intermediate Mechanics for Indian Students. By F. C. TURNER and J. M. BOSE. Pp. 332. Price 4s. 6d. 1914. (Longmans, Green & Co.)

Though the general principle that elementary text-books should not be multiplied beyond necessity may be admitted, there is much to be said for

the text-book designed to meet the wants of a definite class of students. The authors have set themselves the problem of covering the syllabus of the Intermediate Examination at Calcutta University, and furnish an exposition of elementary statics and dynamics, with several illustrations happily chosen with reference to Indian daily life.

The explanatory matter is abundant, and so are the worked-out examples, but unfortunately many of the statements are extremely loose. For instance, at page 47 we read, "Forces P and Q pounds acting in opposite directions on a mass of m pounds produce an acceleration $\frac{P-Q}{m}g$ feet per second per second," a mode of statement which will unite the advocates of the pound, poundal, and slug, until the question of dimensions is raised. And again, at p. 291:—"We conclude therefore that whenever a body moves in any curve it is acted upon by a force along the normal and outwards from the centre." Alas!

A large number of misprints, usually trivial, have escaped the eye of the proof reader, *e.g.* at pp. 78, 80, 117, 119, 120, 169, 211, 225, 231, 232, 258, 261, 262, 263, 264, 283, 294, 299, 313. S. K.

A First Book of Practical Mathematics. By T. S. Usherwood and C. J. A. Trimble. Pp. 183. 1s. 6d. 1913. (Macmillan & Co.)

The book deals in ten chapters with the simple operations of arithmetic and algebra, the use of squared paper, and practical applications of elementary mathematics to simple problems in elementary mensuration and physics. Attempts are made with varying success to justify the mathematical operations involved, but it will be seen from the following quotations that the emphasis is on the *practical* rather than on the *mathematical* side: "The sign $-$ denotes the inverse of the sign $+$." "Division is the inverse operation to multiplication. Thus $a\times\frac{1}{b}$ is the same as $\frac{a}{b}$." "If any numbers whatever be substituted for a, b and x, it will be found that $a(b+x)=ab+ax$." "Drawings will show that equiangular triangles are of the same shape."

The book is well printed, and there is abundance of good examples, many of them taken from examination papers.

Practical Geometry and Graphics for Advanced Students. By Joseph Harrison and G. A. Baxandall. Pp. 677+xiv. 6s. 1913. (Macmillan & Co.)

This book is an enlarged edition of the original work first published in 1899 and subsequently reprinted six times. It covers the syllabus of the Higher Examination of the Board of Education in Practical Geometry and Graphics. The new matter includes (i) a chapter illustrating the application of geometry to the work of the builder, mason, joiner, and metal plate worker, (ii) an extended treatment of graphics, including graphic integration and differentiation, graphic statics in two and three dimensions, and graphic dynamics, (iii) a collection of questions selected from the Board of Education examination papers, grouped and arranged, with answers.

The subject-matter is sub-divided into three sections dealing respectively with Plane Geometry (175 pages), Descriptive Geometry (344 pages), and Graphics (88 pages). Then we have 43 pages of examination questions, 6 pages of mathematical tables, an Appendix enumerating the definitions and theorems of Pure Solid Geometry, and a very full Index.

The book has been carefully and concisely written, the explanations are clear, and the numerous diagrams are well drawn and conveniently arranged. Contrary to the usual practice, the decimal point is invariably placed at the bottom.

The last chapter of Section I. is entitled Plotting on Squared Paper, and is chiefly devoted to the Straight Line. The determination of the equation of a straight line already drawn is based on the calculation of the constants of the equation from the observed coordinates of two points selected near its ends. Would it not be worth while to explain how the gradient can be read off directly from the drawing?

Probably few students will read through the whole of Section II., which includes chapters on Surfaces in Contact, Interpenetration of Solids, Cast Shadows and Metric Projection. It seems a pity that only four pages are devoted to Spherical Triangles.

The perusal of such a book—a goldmine of information on the application of geometry to matters of human interest—makes one desire to bring the fundamental ideas within the reach of schoolboys. Perhaps in time we may hope to see a school course in Solid Geometry based on Motion of Rotation about an axis and Parallel Motion, and developing simultaneously the foundation principles of Pure Geometry, Descriptive Geometry, Coordinate Solid Geometry, and Spherical Trigonometry. W. J. DOBBS.

Nouveaux Essais de Magie arithmétique polygonale. Étoiles magiques à 10 et 12 branches. Pp. 26. **Questions inédites de Magie arithmétique polygonale. Étoiles magiques à 8, 16 et 20 branches et rosaces hypermagiques.** Pp. 21. Par C. SALOMON. 1·50 fr. each. 1913. (Gauthier-Villars, Paris.)

These extensions of a former pamphlet contain many curious and complicated magic figures, all constructed by the same method of superposing two figures with simpler properties. H. P. H.

CORRESPONDENCE.

TO THE EDITOR OF THE *Mathematical Gazette.*

Osborne, 31*st March*, 1914.

SIR,—The interesting argument in your March number between Mr. A. G. Cracknell and your reviewer X. Y. Z. suggests the following reflection: If these questions of fundamental laws in arithmetic and algebra are so difficult as to be a matter of dispute between two competent mathematicians, why trouble beginners about them?

X. Y. Z. "again puts his conundrum: '6 lbs. of sugar $1\frac{1}{2}$ times...?'" Why not 9 lbs.? I don't think that most people would consider it wrong to say that 6 lbs. are contained in 9 lbs. $1\frac{1}{2}$ times, or that the circumference of a circle is about $3\frac{1}{7}$ times the diameter.—Yours faithfully,

CHARLES GODFREY.

THE FORTHCOMING EDINBURGH MATHEMATICAL COLLOQUIUM.

THE successful Colloquium which was held in Edinburgh last August was described in the October number of the *Gazette*, and the suggestion was there made that a Colloquium should be held in the same city in the present year, in conjunction with the Napier Tercentenary Celebrations. This suggestion has been adopted by the Edinburgh Mathematical Society, who have now resolved to hold an open Colloquium on July 28th to 31st inclusive, immediately following the Napier celebrations on July 24th to 27th. The following short courses of lectures have been arranged:

(A) Two lectures by **M. d'Ocagne** (*Professor at the École Polytechnique and the École Nationale des Ponts et Chaussées, Paris, and Past President of the Société Mathématique de France*), on NOMOGRAPHY.

It is now generally recognised that for most purposes the nomographic methods are superior to the older graphical methods of calculation. The introduction of some nomographic teaching in British Universities (and schools, for much of it is not too hard for schoolboys) is much to be desired.

(B) Four lectures by **H. W. Richmond, M.A., F.R.S.** (*Fellow and Lecturer of King's College, Cambridge, and University Lecturer in Mathematics*), on INFINITY IN GEOMETRY.

The "line at infinity," the "cyclic points," and the "circle at infinity" are familiar conceptions, and may serve to indicate the class of questions to be discussed in these lectures.

(C) Four lectures by **E. Cunningham, M.A.** (*Fellow and Lecturer of St. John's College, Cambridge*), on CRITICAL STUDIES OF THE MODERN ELECTRIC THEORIES.

Of recent years most things in Physics have been explained in terms of electrons: and electric theories of the constitution of matter, gravitation, spectroscopy, etc., have been freely produced. Some of these theories will be described, and the points at which they are theoretically incomplete will be indicated.

(D) Two lectures by **E. T. Whittaker, Sc.D., F.R.S.** (*Professor of Mathematics in the University of Edinburgh*), on THE SOLUTION OF ALGEBRAIC AND TRANSCENDENTAL EQUATIONS IN THE MATHEMATICAL LABORATORY.

The methods described will be chiefly arithmetical, and the lectures will therefore be supplementary to the lectures of Professor d'Ocagne on nomographic methods.

The fee for the whole Colloquium is 15s., and should be sent to the Honorary Secretary of the Edinburgh Mathematical Society (P. Comrie, M.A., B.Sc., 19 Craighouse Terrace, Edinburgh). Those who wish also to become ordinary members of the Napier Tercentenary Celebrations may send the fee for this (5s.) at the same time, making a total of £1. An additional fee will be charged to late entries.

In connexion with the Napier Celebrations, an Exhibition of Calculating Apparatus of all kinds, from Napier's Bones to the present time, is being organised, and a descriptive handbook will be issued.

THE LIBRARY.

THE Library has now a home in the rooms of the Teachers' Guild, 74 Gower Street, W.C. A catalogue has been issued to members containing the list of books, etc., belonging to the Association and the regulations under which they may be inspected or borrowed.

The Librarian will gladly receive and acknowledge in the *Gazette* any donation of ancient or modern works on mathematical subjects.

The Librarian acknowledges, with thanks, the receipt of a copy of *A Binary Canon*, by Lt.-Col. Allan Cunningham.

Wanted by purchase or exchange:

1 or 2	copies of *Gazette*	No. 2 (very important).
1 or 2	„ „	No. 8.
2 or 3	copies of Annual Report	No. 11 (very important).
1 or 2	„ „	Nos. 10, 12 (very important).
1 copy	„	Nos. 1, 2.

BOOKS, ETC., RECEIVED.

Annals of Mathematics. Edited by ORMOND STONE and others. March, 1914. Ser. II. Vol. 15. No. 3. 2$ per ann. (Lancaster, Pa. U.S.A.)

On Continued Fractions in Non-Commutative Quantities. J. H. M. WEDDERBURN. *A new Type of Solution of Maxwell's Equations.* H. BATEMAN. *Relation between the Zeros of a Rational Integral Function and its Derivate.* T. HAYASHI. *The Invariants, Semiinvariants and Linear Covariants of the Binary Quartic Form Modulo 3.* L. E. DICKSON. *Examples of Normal Domains of Rationality belonging to Elementary Groups.* G. A. MILLER. *On Lebesgue's Constants in the Theory of Fourier Series.* T. H. GRONWALL. *The Linear Difference Equation of the First Order.* K. P. WILLIAMS. *Geometric Properties of the Jacobians of a Certain System of Functions.* A. EMCH. *On the Irregular Integrals of Linear Differential Equations.* C. E. LOVE.

An Introduction to the Study of Integral Equations. By M. BÔCHER. Pp. 71. 2s. 6d. net. 1914. (Cambridge University Press.) Cambridge Tracts in Mathematics and Mathematical Physics.

Exercices de Géométrie analytique à l'usage des Élèves de Mathématiques spéciales. By P. AUBERT and G. PAPELIER. Pp. 360. 6 frs. 1914. (Vuibert, Paris.)

Higher Algebra. By W. P. MILNE. Pp. 586. 7s. 6d. 1913. (Arnold.)

Mathematical Models. Some Mathematical Booklets. (Bull. Amer. Math. Society. Feb. 1914.) WINSLOW UPTON. (Science, Feb. 6, 1914). By R. C. ARCHIBALD.

Die mathematischen Wissenschaften:—Die Beziehungen der Mathematik zur Kultur der Gegenwart. By A. VOSS. *Die Verbreitung mathematischen Wissen- und mathematischer Auffassung.* Pp. 161. 6 m. 1914. (Teubner.)

A First Year Course in Mathematics, Geometry, and Trigonometry. By K. J. SANJANA. Pp. x+241+xv. 1·12 rupees. 1914. (Cooper, Bombay.)

BOOKS

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